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ON COMPATIBLE DIFFERENTIAL EQUATIONS AND THE ORTHOGONAL PROPERTIES OF THEIR SOLUTIONS

By W. O. KERMACK (*Edinburgh*) AND W. H. McCREA (*London*)

[Received 2 December 1932]

1. Introduction. In some previous work† we have defined conjugate partial differential equations for a function of two variables. In the present paper we begin by examining more fully the condition for their compatibility and that of their adjoint equations. We then extend the work to the case where, instead of partial derivatives with respect to one of the variables, we have *differences* with respect to that variable. This is the case applicable to discrete eigenvalues. It is well known that the eigensolutions of a linear differential equation involving a parameter are orthogonal to those of its adjoint equation. This parameter can be treated as the second variable in the present work, and a difference equation with respect to its eigenvalues can be written down for the eigenfunctions. What we now show is that the eigensolutions of this differential and difference equation, and those of the adjoint pair of equations, form *normalized* orthogonal sets.

We do not examine fully the general conditions under which the latter theorem is true. Our purpose is rather to show how it can be used to predict the existence and form of orthogonality relations, the strict validity of which may require independent investigation. Treated in this way the work brings out a fresh aspect of the relation between the differential and difference equations satisfied by functions of hypergeometric type, and the relation of these equations to the orthogonal property.

Although the present paper is more or less self-contained, we retain the notation of previous papers, since we hope elsewhere to develop the theory from the point of view of the algebra of operators.

2. Conjugate differential equations. We denote by q, Q two independent variables, and by p, P the operators $-\partial/\partial q$, $\partial/\partial Q$

† W. O. Kermack and W. H. McCrea, *Proc. Edinburgh Math. Soc.* (2) 2 (1931), 205–19, 220–39; *Proc. Roy. Soc. Edinburgh*, 41 (1931), 176–89. These will be referred to as Papers I, II, III. Conjugate equations are defined in Paper II.

respectively. A pair of partial differential equations for the function $\chi(q, Q)$ of the form

$$[Q(q,p)-Q]\chi = 0 \tag{2.01}$$

$$[P(q,p)-P]\chi = 0, \tag{2.02}$$

where $Q(q,p)$, $P(q,p)$ are given functions of q, p, will be called *conjugate equations* in *normal form*. We have

THEOREM I. *A necessary and sufficient condition that the equations* (2.01), (2.02) *should be compatible is that*

$$Q(q,p)P(q,p)-P(q,p)Q(q,p) = 1 \tag{2.03}$$

identically, where $qp-pq = 1$.

This was given as Theorem XIII of Paper II. In order, however, to make explicit the degree of arbitrariness of the common solution $\chi(q, Q)$, we shall give an independent proof of the sufficiency of the condition (2.03). This amounts, as a matter of fact, to finding of how many independent interpretations the function $\varpi(q,p)$ of Paper II is susceptible, when treated as an operator. We hope later to consider this problem directly. We now show that

THEOREM II. *If the condition* (2.03) *is satisfied, and if* $Q(q,p)$ *is an integral function of* p *of degree* n, *then there are in general* n, *and only* n, *common solutions* $\chi(q,p)$ *of* (2.01), (2.02), *linearly independent in both* q *and* Q.

We shall write out the proof for $n = 3$, but the method is general.

Equation (2.01) by itself is a linear ordinary differential equation for χ as a function of q, and will possess three solutions linearly independent in q. These we denote by $\chi_1(q, Q)$, $\chi_2(q, Q)$, $\chi_3(q, Q)$; so that the general solution of (2.01) is

$$\chi = a_1\chi_1+a_2\chi_2+a_3\chi_3, \tag{2.04}$$

where a_1, a_2, a_3 are arbitrary functions of Q.

Since
$$[Q(q,p)-Q]\chi_1 = 0, \tag{2.05}$$

then
$$[P(q,p)Q(q,p)-P(q,p)Q]\chi_1 = 0$$

or, using (2.03),

$$[Q(q,p)P(q,p)-QP(q,p)-1]\chi_1 = 0. \tag{2.06}$$

Similarly we have again, from (2.05),

$$[PQ(q,p)-PQ]\chi_1 = 0$$

or, since $PQ-QP = 1$,

$$[Q(q,p)P-QP-1]\chi_1 = 0. \tag{2.07}$$

Hence, subtracting (2.06), (2.07), we have

$$[\mathrm{Q}(q,p)-Q][\mathrm{P}(q,p)-P]\chi_1 = 0. \tag{2.08}$$

Therefore $[\mathrm{P}(q,p)-P]\chi_1$ is itself a solution of (2.01), and so must be of the form (2.04). This we may express, together with the corresponding results for χ_2, χ_3, by saying that functions a_{11}, a_{12},..., a_{33} of Q exist, such that

$$[\mathrm{P}(q,p)-P]\chi_1 = a_{11}\chi_1+a_{12}\chi_2+a_{13}\chi_3, \tag{2.09}$$

$$[\mathrm{P}(q,p)-P]\chi_2 = a_{21}\chi_1+a_{22}\chi_2+a_{23}\chi_3, \tag{2.10}$$

$$[\mathrm{P}(q,p)-P]\chi_3 = a_{31}\chi_1+a_{32}\chi_2+a_{33}\chi_3. \tag{2.11}$$

Multiplying these equations by functions $b_1(Q)$, $b_2(Q)$, $b_3(Q)$ respectively, and adding, we obtain

$$\begin{aligned}[\mathrm{P}(q,p)-P]\chi = {} & (a_{11}b_1-b_1'+a_{21}b_2+a_{31}b_3)\chi_1+ \\ & +(a_{12}b_1+a_{22}b_2-b_2'+a_{32}b_3)\chi_2+ \\ & +(a_{13}b_1+a_{23}b_2+a_{33}b_3-b_3')\chi_3,\end{aligned} \tag{2.12}$$

where $b_1' = db_1/dQ$, etc., and where

$$\chi = b_1\chi_1+b_2\chi_2+b_3\chi_3. \tag{2.13}$$

Since χ_1, χ_2, χ_3 are linearly independent, it follows from (2.12) that χ is a solution of (2.02), that is,

$$[\mathrm{P}(q,p)-P]\chi = 0,$$

if and only if the coefficients of χ_1, χ_2, χ_3 on the right-hand side of (2.12) vanish separately, giving

$$a_{11}b_1-b_1'+a_{21}b_2+a_{31}b_3 = 0, \tag{2.14}$$

$$a_{12}b_1+a_{22}b_2-b_2'+a_{32}b_3 = 0, \tag{2.15}$$

$$a_{13}b_1+a_{23}b_2+a_{33}b_3-b_3' = 0. \tag{2.16}$$

These equations determine the functions b_1, b_2, b_3. The elimination of any two of them yields in general a linear differential equation of order three in the remaining one. To each of its three solutions there will then correspond one set of functions b_1, b_2, b_3. Hence, in general, there are three, and only three, functions of the type χ, and the theorem is proved. Any linear combination of these functions with *constant* coefficients will be a common solution of (2.01) and (2.02).

3. Adjoint equations. We define the *adjoint* $f^*(q,p)$ of a function $f(q,p)$ of q,p as the function obtained by reversing the order of the factors in each term of $f(q,p)$, and at the same time changing the sign of p. Thus, for example, if $f(q,p) = q^2p^2+qp+1$, then

$f^*(q,p) = p^2q^2 - pq + 1$. In particular, the adjoint of a product of two functions $f(q,p)$, $g(q,p)$ is given by

$$(fg)^* = g^*f^*.$$

The *adjoint* of a given differential equation is then defined as the equation obtained from it by replacing all the functions of operators occurring by the adjoint functions.

Now it has been shown by N. H. McCoy† that the adjoint of an identity is itself an identity. Applying this to the identity (2.03), we have

$$\mathrm{P}^*(q,p)\mathrm{Q}^*(q,p) - \mathrm{Q}^*(q,p)\mathrm{P}^*(q,p) = 1. \tag{3.1}$$

But from Theorem I this is precisely the necessary and sufficient condition that the pair of differential equations

$$[\mathrm{Q}^*(q,p) - Q]\chi^* = 0 \tag{3.2}$$

$$[\mathrm{P}^*(q,p) + P]\chi^* = 0 \tag{3.3}$$

for the function $\chi^*(q,Q)$ should be compatible. Now these are the equations adjoint to (2.01), (2.02). Thus we have shown

THEOREM III. *The adjoint equations of a pair of compatible differential equations in normal form are themselves compatible and in normal form.*

Clearly the number of independent solutions of (3.2), (3.3) is given by Theorem II, and is the same as the number of solutions of (2.01), (2.02). For if $\mathrm{Q}(q,p)$ is of degree n in p, then $\mathrm{Q}^*(q,p)$ is also of degree n in p.

4. Second type of compatible equations. We shall be led by later applications to consider a pair of equations for the function $\chi(q,Q)$ of the form

$$[\mathrm{Q}(q,p) - Q]\chi = 0 \tag{4.1}$$

$$[\mathrm{S}(q,p) - e^P]\chi = 0, \tag{4.2}$$

where $\mathrm{Q}(q,p)$, $\mathrm{S}(q,p)$ are given functions of q,p, and where we write formally for any function of Q, $f(Q+1) = e^P f(Q)$. Hence (4.2) is actually a *difference* equation for χ.

The following results can now be proved by methods exactly similar to those already used for the corresponding theorems in §§ 3, 4.

THEOREM IV. *A necessary and sufficient condition that equations* (4.1), (4.2) *should be compatible is*

$$\mathrm{Q}(q,p)\mathrm{S}(q,p) - \mathrm{S}(q,p)\mathrm{Q}(q,p) = \mathrm{S}(q,p) \tag{4.3}$$

identically, where $qp - pq = 1$.

† N. H. McCoy, *Trans. American Math. Soc.* 31 (1929), 793–806, Theorem II.

THEOREM V. *If the identity* (4.3) *is satisfied, and if* $Q(q,p)$ *is an integral function of* p *of degree* n, *then there are* n, *and only* n, *common solutions of* (4.1), (4.2), *linearly independent in both* q *and* Q.

The proof of Theorem V follows the steps of Theorem I, but leads to a *difference* equation of order n for each of the functions $b_1, b_2, \ldots$. The n linearly independent solutions of this difference equation can be multiplied by arbitrary functions of Q of unit period. But since we shall in all applications confine attention to values of Q at unit interval, these periodic functions have no effect, and can be ignored from the outset.

THEOREM VI. *The identity* (4.3) *gives also a necessary and sufficient condition that the pair of equations adjoint to* (4.1), (4.2) *should be compatible.*

The adjoint equations are

$$[Q^*(q,p)-Q]\chi^* = 0 \tag{4.4}$$

$$[S^*(q,p)-e^{-P}]\chi^* = 0. \tag{4.5}$$

If they are compatible they possess n solutions linearly independent in both q and Q.

5. Orthogonal properties. We shall now show

THEOREM VII. *If a pair of compatible difference-differential equations are given in the form*

$$[Q(q,p)-Q]\chi = 0, \tag{5.01}$$

$$[S(q,p)-e^{P}]\chi = 0, \tag{5.02}$$

and their adjoint equations in the form

$$[Q^*(q,p)-Q]\chi^* = 0, \tag{5.03}$$

$$[S^*(q,p)-e^{-P}]\chi^* = 0, \tag{5.04}$$

and if a pair of solutions $\chi(q,Q)$, $\chi^*(q,Q)$ *can be made to satisfy suitable boundary conditions at end-points* $q=a$, $q=b$ (*say*), *by restricting* Q *to assume integral eigenvalues, then the functions* χ, χ^* *are normalized orthogonal functions with respect to the interval* (a,b), *i.e.*

$$\int_a^b \chi(q,Q)\chi^*(q,Q')\,dq = k\delta_{QQ'}, \tag{5.05}$$

where $\delta_{QQ'} = 0$, $Q \neq Q'$; $\delta_{QQ} = 1$, *and where* k *is a purely numerical constant independent of* Q.

Suppose $Q(q,p)$ is expanded as a sum of terms of the form q^ip^j where, since we still assume $Q(q,p)$ to be an integral function of p,

j must be a positive integer. Consider, then, the integral

$$\int_a^b \{q^i p^j \chi\}\chi^* \, dq$$

$$= [-p^{j-1}\chi \cdot q^i\chi^* - p^{j-2}\chi \cdot (-p)q^i\chi^* - \ldots - \chi \cdot (-p)^{j-1}q^i\chi^*]_a^b + \\ + \int_a^b \chi\{(-p)^j q^i \chi^*\} \, dq \tag{5.06}$$

$$= [\ldots]_a^b + \int_a^b \chi\{(q^i p^j)^* \chi^*\} \, dq \tag{5.07}$$

by elementary integration by parts. Treating each term of $Q(q, p)$ in this way and adding the results, we have, using (5.01) and (5.03),

$$Q \int_a^b \chi(q, Q)\chi^*(q, Q') \, dq = \int_a^b \{Q(q, p)\chi(q, Q)\}\chi^*(q, Q') \, dq$$

$$= \int_a^b \chi(q, Q)\{Q^*(q, p)\chi^*(q, Q')\} \, dq$$

$$= Q' \int_a^b \chi(q, Q)\chi^*(q, Q') \, dq, \tag{5.08}$$

together with an integrated part, given by the sum of terms corresponding to those in square brackets in (5.06).

The condition that this integrated part should vanish for the endpoints a, b, for all permitted values of Q, Q' provides the eigenvalue problem. This may or may not have a solution. In a certain class of cases it is solved by taking integral eigenvalues for Q; and we confine ourselves for the moment to such cases.

Hence we have from (5.08) for the corresponding eigenfunctions $(Q - Q')\psi = 0$, where

$$\psi(Q, Q') = \int_a^b \chi(q, Q)\chi^*(q, Q') \, dq. \tag{5.09}$$

Using equations (5.02), (5.04), and proceeding as in the derivation of (5.08), we have similarly

$$e^P \int_a^b \chi(q, Q)\chi^*(q, Q') \, dq = e^{-P'} \int_a^b \chi(q, Q)\chi^*(q, Q') \, dq, \tag{5.10}$$

together with an integrated part, where P' is conjugate to Q'. Now we shall assume that we are dealing with cases where *the same boundary conditions that cause the integrated part of* (5.08) *to vanish*

cause also the integrated part of (5.10) *to vanish.* The examples we shall give serve to show that this holds in a large class of cases, but we shall not here investigate the general conditions of its validity. Hence from (5.10)

$$(e^P - e^{-P})\psi = 0, \text{ or } \psi(Q+1, Q') - \psi(Q, Q'-1) = 0. \quad (5.11)$$

Now (5.09) expresses the usual result for eigenfunctions, namely,

$$\psi(Q, Q') = K(Q)\delta_{QQ'}, \quad (5.12)$$

where $K(Q)$ is some function of Q (provided the integral has no singularity at $Q = Q'$).

Writing $Q' = Q+1$ in (5.11) and using (5.12), we now have

$$K(Q+1) = K(Q) = k, \text{ say}, \quad (5.13)$$

where k is a constant independent of Q. This is the required result.

The orthogonal property of the eigensolutions of a single linear differential equation and its adjoint equation are, of course, well known. The extension here introduced is in showing that the eigensolutions of a compatible pair of equations, and the adjoint pair, are orthogonal *and normalized.*

6. More general form of the equations. A more general form covering all the examples we shall consider is given by a pair of equations

$$[f(q, p) - F(Q, e^P)]\chi = 0, \quad (6.01)$$

$$[g(q, p) - G(Q, e^P)]\chi = 0, \quad (6.02)$$

where f, g, F, G are given functions, and where the operators have no common factors.

We shall not investigate a sufficient condition for the compatibility of a pair of equations of this type. But we shall suppose that a common solution $\chi(q, Q)$ is known to exist. Then if this $\chi(q, Q)$ were given, we could by the methods described in our Paper I derive a pair of conjugate equations formally satisfied by χ, and by a further application of the same formal methods we can transform these into the type

$$[\mathrm{Q}(q, p) - Q]\chi = 0, \qquad [\mathrm{S}(q, p) - e^P]\chi = 0. \quad (6.03)$$

Since, then, χ is a solution of (6.01), (6.02), we have by substituting the values of $Q\chi$, $e^P\chi$ given by (6.03), the relations

$$f(q, p) - \widetilde{F}\{\mathrm{Q}(q, p), \mathrm{S}(q, p)\} = 0 \quad (6.04)$$

$$g(q, p) - \widetilde{G}\{\mathrm{Q}(q, p), \mathrm{S}(q, p)\} = 0 \quad (6.05)$$

satisfied identically, where $qp - pq = 1$, and where $\widetilde{F}$, $\widetilde{G}$ are the

transposed functions F, G obtained by reversing the order of the factors in every term of F, G.

By McCoy's theorem already quoted we have now from (6.04), (6.05) the adjoint identities

$$f^*(q,p) - \tilde{F}^*\{Q^*(q,p), S^*(q,p)\} = 0, \tag{6.06}$$

$$g^*(q,p) - \tilde{G}^*\{Q^*(q,p), S^*(q,p)\} = 0. \tag{6.07}$$

Therefore any solution of the equation

$$[Q^*(q,p) - Q]\chi^* = 0, \qquad [S^*(q,p) - e^{-P}]\chi^* = 0 \tag{6.08}$$

is also a solution of the equations†

$$[f^*(q,p) - F^*(Q,e^P)]\chi^* = 0, \tag{6.09}$$

$$[g^*(q,p) - G^*(Q,e^P)]\chi^* = 0. \tag{6.10}$$

But since equations (6.03) are by their derivation compatible and have the solution χ, it follows by Theorem VI that equations (6.08) are compatible and have a common solution χ^*. Hence we have shown that the equations (6.09), (6.10) adjoint to equations (6.01), (6.02) are compatible, and have the common solution χ^*. Further, provided suitable boundary conditions can be satisfied by taking integral eigenvalues for Q, then by Theorem VII applied to (6.03), (6.08), the functions χ, χ^* are orthogonal and normalized. So we have

THEOREM VIII. *If a pair of distinct difference-differential equations*

$$[f(q,p) - F(Q,e^P)]\chi = 0, \qquad [g(q,p) - G(Q,e^P)]\chi = 0$$

possess a common solution $\chi(q,Q)$, then the adjoint equations

$$[f^*(q,p) - F^*(Q,e^P)]\chi^* = 0, \qquad [g^*(q,p) - G^*(Q,e^P)]\chi^* = 0$$

possess a common solution $\chi^*(q,Q)$, and, provided suitable boundary conditions at end-points $q = a$, $q = b$ (say) are satisfied by taking integral eigenvalues for Q, we have further

$$\int_a^b \chi(q,Q)\chi^*(q,Q')\,dq = k\delta_{QQ'},$$

where k is a constant independent of Q.

7. Non-integral eigenvalues. If the boundary-value problem leads to non-integral eigenvalues for Q, we can still apply the theory, provided the eigenvalues Q are expressible as functions of an integral variable N. For we can then change the variable from Q to N and

† Note that it is correct to write $F^*(Q,e^P)$, and not $F^*(Q,e^{-P})$. For example, if $F(Q,e^P) = Qe^P$, then $F^*(Q,e^P) = e^{-P}Q$ gives the adjoint function.

equations of the form (6.01), (6.02) will transform into others of the same type. These will have integral eigenvalues and Theorem VIII applies.

8. Continuous eigenvalues. A similar theory can be written down *formally* for the case of continuous eigenvalues, with Dirac's δ-function replacing the Kronecker δ-symbol. But the satisfaction of the boundary conditions appears to be more difficult, and we have not succeeded in constructing examples for which it holds.

9. Applications and examples. As an example of a pair of conjugate equations in normal form we may take the pair

$$(p^2+Q)\chi = 0, \qquad (\tfrac{1}{2}qp^{-1}-P)\chi = 0.$$

Here we have $\mathrm{Q}(q,p) = -p^2$, $\mathrm{P}(q,p) = \tfrac{1}{2}qp^{-1}$, so that

$$\mathrm{Q}(q,p)\mathrm{P}(q,p)-\mathrm{P}(q,p)\mathrm{Q}(q,p) = -\tfrac{1}{2}p^2qp^{-1}+\tfrac{1}{2}qp^{-1}p^2 = 1,$$

and therefore the identity (2.03) is satisfied.

We find by direct solution that there are two, and only two, linearly independent solutions

$$\chi = \sin q\sqrt{Q}, \qquad \chi = \cos q\sqrt{Q},$$

illustrating the result of Theorem II.

The adjoint equations are

$$(p^2+Q)\chi^* = 0, \qquad (\tfrac{1}{2}p^{-1}q+P)\chi^* = 0.$$

We find again by direct solution that these possess just two independent solutions

$$\chi^* = \frac{1}{\sqrt{Q}}\sin q\sqrt{Q}, \qquad \chi^* = \frac{1}{\sqrt{Q}}\cos q\sqrt{Q}.$$

This verifies Theorem III.

10. Legendre polynomials $P_n(z)$. The usual differential and difference equations for $P_n(z)$ are

$$\left[(1-z^2)\frac{d^2}{dz^2}-2z\frac{d}{dz}+n(n+1)\right]P_n(z) = 0 \tag{10.1}$$

$$(n+1)P_{n+1}(z)-(2n+1)zP_n(z)+nP_{n-1}(z) = 0. \tag{10.2}$$

To adhere to our customary notation we shall write $P_n(z)$ as $\chi(q, Q)$, where q, Q replace z, n respectively. Then (10.1), (10.2) become

$$[p(1-q^2)p+Q(Q+1)]\chi = 0 \tag{10.3}$$

$$\left[q-\frac{Q+1}{2Q+1}e^P-\frac{Q}{2Q+1}e^{-P}\right]\chi = 0. \tag{10.4}$$

These equations are by their derivation compatible, and we can apply Theorem VIII.

The adjoint equations are

$$[p(1-q^2)p+Q(Q+1)]\chi^* = 0 \tag{10.5}$$

$$\left[q-e^{-P}\frac{Q+1}{2Q+1}-e^{P}\frac{Q}{2Q+1}\right]\chi^* = 0. \tag{10.6}$$

Since the first equation is self-adjoint and does not involve the operator P, we try a solution

$$\chi^*(q, Q) = \beta(Q)\chi(q, Q), \tag{10.7}$$

where $\beta(Q)$ is a function of Q only. We find that this gives a solution of (10.6) if β satisfies the equation

$$\frac{\beta(Q+1)}{2Q+3} = \frac{\beta(Q)}{2Q+1} = \frac{\beta(Q-1)}{2Q-1}.$$

That is, we may take $\beta(Q) = 2Q+1$.

Thus the *compatibility of the adjoint equations* is verified. Theorem VIII will further give the orthogonal property

$$\int_a^b \chi(q, Q)\chi^*(q, Q')\,dq = k\delta_{QQ'}, \tag{10.8}$$

provided suitable end-points a, b can be found. But it is clear from the form of the differential equations (10.3), (10.5) that we may take $a = -1$, $b = 1$. If, then, we fix k so that $P_0(z) = 1$, we must have

$$\int_{-1}^{1} dq = k, \text{ or } k = 2.$$

Returning now to the familiar notation, and writing n, m for Q, Q', we see that (10.8) is the familiar result

$$\int_{-1}^{1} P_n(z)P_m(z)\,dz = \frac{2\delta_{mn}}{2n+1}.$$

We have thus derived the orthogonal property of the Legendre polynomials directly from the differential and difference equations.†

† This should be compared with the work of Ferrar, *Proc. Edinburgh Math. Soc.* (2) 2 (1930), 71. He establishes a connexion between the orthogonality and the differential equation, but scarcely such a simple one as the present, since he does not keep the differential and difference equations together as a conjugate pair.

11. Parabolic cylinder functions $D_n(z)$. These† satisfy Weber's differential equation

$$\left(\frac{d^2}{dz^2}+n+\tfrac{1}{2}-\tfrac{1}{4}z^2\right)D_n(z)=0 \tag{11.1}$$

and the recurrence relation

$$D_n'(z)-\tfrac{1}{2}zD_n(z)+D_{n+1}(z)=0. \tag{11.2}$$

Translated into operational symbols, with q, Q for z, n respectively, these equations become

$$(p^2-\tfrac{1}{4}q^2+\tfrac{1}{2}+Q)\chi=0, \tag{11.3}$$

$$(p+\tfrac{1}{2}q-e^P)\chi=0. \tag{11.4}$$

The adjoint equations are, then,

$$(p^2-\tfrac{1}{4}q^2+\tfrac{1}{2}+Q)\chi^*=0, \tag{11.5}$$

$$(p-\tfrac{1}{2}q+e^{-P})\chi^*=0. \tag{11.6}$$

It is evident from the self-adjointness of (11.3) that we may take $\chi^*=\gamma(Q)\chi$, where $\gamma(Q)$ is a function of Q alone. It is then not difficult to show that $\gamma(Q)$ must satisfy the recurrence relation

$$Q\gamma(Q)=\gamma(Q-1).$$

Hence we may take $\gamma(Q)=1/\Gamma(Q+1)$.

The general theorem, Theorem VII, indicates therefore that $D_n(z)$ and $(1/n!)D_n(z)$ are orthogonal functions. So we must have

$$\int_a^b D_n(z)D_m(z)\,dz=kn!\delta_{nm}, \tag{11.7}$$

provided suitable end-points a, b can be found. But since we have $D_0(z)=e^{-\frac{1}{4}z^2}$, we may compare (11.7) with

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2}\,dz=\sqrt{2\pi}.$$

Therefore the precise form of (11.7) is the well-known result

$$\int_{-\infty}^{\infty} D_n(z)D_m(z)\,dz=\sqrt{2\pi}\,n!$$

12. Bessel coefficients $J_n(z)$. These satisfy Bessel's differential equation

$$\left(\frac{d^2}{dz^2}+\frac{1}{z}\frac{d}{dz}+\left(1-\frac{n^2}{z^2}\right)\right)J_n(z)=0 \tag{12.1}$$

† Whittaker and Watson, *Modern Analysis*, 347. (1927.)

and the recurrence relation

$$J_{n-1}(z)+J_{n+1}(z) = (2n/z)J_n(z). \tag{12.2}$$

These we shall write in the equivalent forms, with q, Q for z, n respectively,

$$(qpqp+q^2-Q^2)\chi = 0, \tag{12.3}$$

$$\{2q^{-1}-Q^{-1}(e^P+e^{-P})\}\chi = 0. \tag{12.4}$$

The adjoint equations are

$$(pqpq+q^2-Q^2)\chi^* = 0, \tag{12.5}$$

$$\{2q^{-1}-(e^P+e^{-P})Q^{-1}\}\chi^* = 0. \tag{12.6}$$

It is immediately clear that (12.5) is satisfied by

$$\chi^*(q, Q) = q^{-1}\sigma(Q)\chi(q, Q),$$

where $\sigma(Q)$ is a function of Q alone, since this transforms (12.5) into (12.3). This will further give a solution of (12.6) if

$$\{2q^{-1}\sigma(Q)-(Q+1)^{-1}\sigma(Q+1)e^P-(Q-1)^{-1}\sigma(Q-1)e^{-P}\}\chi = 0$$

is equivalent to (12.4). This is the case provided

$$\frac{\sigma(Q-1)}{Q-1} = \frac{\sigma(Q)}{Q} = \frac{\sigma(Q+1)}{Q+1}.$$

So we may take $\sigma(Q) = Q$.

Theorem VIII then shows that χ, χ^* are orthogonal and normalized provided suitable eigenvalues of Q exist for a suitable range of integration. Thus it suggests the existence of a relation of the form, when we revert to the more usual notation,

$$\int_a^b (n/z)J_n(z)J_m(z)\,dz = k\delta_{mn}. \tag{12.7}$$

Actually there does not seem to be any way of getting the precise relation without recourse to the ordinary theory. This gives it as†

$$\int_0^\infty J_{2m+1}(z)J_{2n+1}(z)\frac{dz}{z} = \frac{\delta_{mn}}{2(2n+1)}. \tag{12.8}$$

So the eigenvalues of Q appropriate to the problem are the odd integral values.

This is a case where our theory fails to give the details of the orthogonality relation. Nevertheless it does suggest the existence and general form of a relation which the detailed theory must seek to establish.

† Watson, *Theory of Bessel Functions*, 404. (1922.)

ON THE SUMMABILITY OF DERIVED SERIES OF THE FOURIER-LEBESGUE TYPE

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1. Introduction.† In two recent papers‡ L. S. Bosanquet and E. H. Linfoot introduced a method of summability which is a generalization of Riesz's arithmetic means of order α and is defined as follows. The series $\sum a_\nu$ is said to be summable (α, β) to S, where either $\alpha > 0$, or $\alpha = 0$, $\beta \geqslant 0$, if

$$\sum_{\nu<n}\left[B\left(1-\frac{\nu}{n}\right)^{\alpha}\log^{-\beta}\left(\frac{C}{1-\nu/n}\right)a_\nu\right] \to S, \quad \text{as} \quad n \to \infty, \qquad (1.01)$$

for C sufficiently large§, where $B = (\log C)^\beta$. When $\alpha = 0$, $\beta \geqslant 0$, the method has been called zero order summability.

Throughout this paper the function $f(x)$ will be understood to be periodic and of period 2π. We shall assume that it is integrable in the sense of Lebesgue, but in the latter part of the discussion (§§ 4–6) restrict it to be of bounded variation in $(0, 2\pi)$. Its rth derived Fourier series is defined to be

$$\sum_{\nu=1}^{\infty} i^r[\nu^r(a_\nu \cos \nu x + b_\nu \sin \nu x)] \qquad (r \text{ even}) \qquad (1.02)$$

and

$$\sum_{\nu=1}^{\infty} -i^{r+1}[\nu^r(-a_\nu \sin \nu x + b_\nu \cos \nu x)] \qquad (r \text{ odd}), \qquad (1.03)$$

where a_ν and b_ν are the Fourier coefficients. In particular the first derived Fourier series and its conjugate series are given by

$$\sum_{\nu=1}^{\infty} [\nu(-a_\nu \sin \nu x + b_\nu \cos \nu x)] \qquad (1.04)$$

and

$$\sum_{\nu=1}^{\infty} [\nu(a_\nu \cos \nu x + b_\nu \sin \nu x)] \qquad (1.05)$$

respectively.

If

$$\frac{f(x+t)+f(x-t)}{2} = f^{(0)}(x) + \frac{f^{(2)}(x)t^2}{2!} + \ldots + \frac{f^{(2r-2)}(x)t^{2r-2}}{(2r-2)!} + \\ + \frac{f^{(2r)}(x)t^{2r}}{(2r)!} + \frac{\omega_1(x,t)t^{2r}}{(2r)!}, \qquad (1.06)$$

† The author is indebted to Professor J. D. Tamarkin for many helpful suggestions.

‡ Bosanquet and Linfoot (**1**), (**2**).

§ Bosanquet and Linfoot (**2**). They have shown that it is equivalent to say 'for every $C > 1$'.

where $f^{(0)}(x), f^{(2)}(x), \ldots, f^{(2r)}(x)$ are independent of t, and $\lim_{t\to 0} \omega_1(x,t) = 0$, then $f^{(2r)}(x)$ is defined to be the generalized derivative of order $2r$ of $f(x)$ at the point x.† If

$$\frac{f(x+t)-f(x-t)}{2} = \frac{f^{(1)}(x)t}{1!} + \frac{f^{(3)}(x)t^3}{3!} + \ldots + \frac{f^{(2r-1)}(x)t^{2r-1}}{(2r-1)!} + \\ + \frac{f^{(2r+1)}(x)t^{2r+1}}{(2r+1)!} + \frac{\omega_2(x,t)t^{2r+1}}{(2r+1)!}, \quad (1.07)$$

where $f^{(1)}(x), f^{(3)}(x), \ldots, f^{(2r+1)}(x)$ are independent of t, and

$$\lim_{t\to 0} \omega_2(x,t) = 0,$$

then $f^{(2r+1)}(x)$ is defined to be the generalized derivative of order $2r+1$ of $f(x)$ at the point x.†

The Cesàro summability of the Fourier series, its conjugate series, and the series defined by (1.02), (1.03), (1.04), and (1.05) has been investigated by various writers. The aim of this paper is to investigate the summability, according to the method defined by Bosanquet and Linfoot, of the series (1.02), (1.03), (1.04), and (1.05). The main results are as follows.

THEOREM 3.1. *The rth derived series of the Fourier series of a function $f(x)$, integrable-L, is summable (α,β), where $\alpha = r$, $\beta > 1$, to the rth generalized derivative whenever it exists.*

THEOREM 5.1. *The derived series of the Fourier series of a function $f(x)$ of bounded variation is summable $(0,\beta)$ almost everywhere, for $\beta > 1$, to $f'(x)$.*

THEOREM 6.2. *The series conjugate to the derived series of the Fourier series of a function $f(x)$ of bounded variation is summable $(0,\beta)$, for $\beta > 1$, to*

$$T \equiv \lim_{\eta\to 0} -\frac{1}{\pi}\int_{\eta}^{\infty} \frac{\psi(t)}{t^2}\,dt$$

almost everywhere, where $\psi(t)$ is defined by (4.01).

2. We shall first define the following functions:

$$H_{l,\alpha,\beta}(1-u) = Bu^{l}(1-u)^{\alpha-1}\log^{-\beta}\left(\frac{C}{1-u}\right) \quad (2.01)$$

for $l \geqslant 0$, $\alpha \geqslant 1$, $\beta \geqslant 0$, where $B = (\log C)^{\beta}$,

$$Q_{l,\alpha,\beta}(t) = \int_0^1 H_{l,\alpha,\beta}(1-u)\cos tu\,du, \quad (2.02)$$

† de la Vallée Poussin (3), 214.

$$\bar{Q}_{l,\alpha,\beta}(t) = \int_0^1 H_{l,\alpha,\beta}(1-u)\sin tu\, du, \tag{2.03}$$

$$M(t) = Q_{l,\alpha,\beta}(t)+i\bar{Q}_{l,\alpha,\beta}(t) = \int_0^1 H_{l,\alpha,\beta}(1-u)e^{itu}\, du, \tag{2.04}$$

$$\lambda_{\alpha,\beta}(n,t) = \frac{1}{\pi}\int_0^n H_{0,\alpha,\beta}\left(1-\frac{v}{n}\right)\cos vt\, dv, \tag{2.05}$$

and $\lambda^{(k)}_{\alpha,\beta}(n,t)$ will denote the kth derivative of $\lambda_{\alpha,\beta}(n,t)$ with respect to t. We shall understand throughout that K, either with or without subscripts, denotes a positive numerical constant.

LEMMA 2.1. *For $l \geqslant 0$, $\alpha \geqslant 1$, $\beta \geqslant 0$, the functions $Q_{l,\alpha,\beta}(t)$ and $\bar{Q}_{l,\alpha,\beta}(t)$ are bounded in $(0,\infty)$, and for large values of t*

$$M(t) = \frac{i^{l+1}l!}{t^{l+1}} + \frac{i^{l+2}c_{l+2}}{t^{l+2}} + O\left(\frac{1}{t^{l+3}}\right) + O\left(\frac{1}{t^{\alpha}\log^{\beta}t}\right), \tag{2.06}$$

where c_{l+2} is a constant.

Proof. The first part of the lemma follows immediately as a consequence of the definitions of $Q_{l,\alpha,\beta}(t)$ and $\bar{Q}_{l,\alpha,\beta}(t)$. The second part is obtained by using Cauchy's theorem to express M in terms of contour integrals, taken round the rectangle of which the vertices are 0, 1, $1+iR$, and iR, letting $R \to \infty$ to derive the expression

$$M(t) = \int_0^{i\infty} H_{l,\alpha,\beta}(1-u)e^{itu}\, du - \int_1^{1+i\infty} H_{l,\alpha,\beta}(1-u)e^{itu}\, du$$

and investigating these integrals independently.†

LEMMA 2.2. *$\lambda^{(k)}_{\alpha,\beta}(n,t)$ is bounded in $(0,\infty)$ for fixed n, where $k \geqslant 0$, $\alpha \geqslant 1$, $\beta \geqslant 0$; and for large values of t*

$$\lambda^{(k)}_{\alpha,\beta}(n,t) = O\left(\frac{1}{t^{k+1}\log^{\beta}nt}\right) \qquad (\alpha = k+1),$$

$$= O\left(\frac{1}{nt^{k+2}}\right) \qquad (\alpha \geqslant k+2),$$

where $k \geqslant 0$ and $\beta \geqslant 0$.

Proof. The boundedness of $\lambda^{(k)}_{\alpha,\beta}(n,t)$ follows from its definition. It

† Cf. Bosanquet and Linfoot (2).

is easily verified that

$$\lambda^{(k)}_{\alpha,\beta}(n,t) = \frac{i^k n^{k+1}}{\pi} Q_{k,\alpha,\beta}(nt) \qquad (k \text{ even}),$$

$$= \frac{i^{k+1} n^{k+1}}{\pi} \bar{Q}_{k,\alpha,\beta}(nt) \qquad (k \text{ odd}).$$

The lemma now follows from (2.06).

LEMMA 2.3. *When* $\alpha > 1$, $\beta \geqslant 0$, $r \geqslant 0$,

$$2\int_0^\infty \lambda^{(2r)}_{\alpha,\beta}(n,t)\cos \nu t\, dt = Bi^{2r}\left(1-\frac{\nu}{n}\right)^{\alpha-1} \log^{-\beta}\left(\frac{C}{1-\nu/n}\right)\nu^{2r} \qquad (0 \leqslant \nu \leqslant n),$$

$$= 0 \qquad (\nu > n);$$

and

$$2\int_0^\infty \lambda^{(2r+1)}_{\alpha,\beta}(n,t)\sin \nu t\, dt$$

$$= Bi^{2r+2}\left(1-\frac{\nu}{n}\right)^{\alpha-1} \log^{-\beta}\left(\frac{C}{1-\nu/n}\right)\nu^{2r+1} \qquad (0 \leqslant \nu \leqslant n),$$

$$= 0 \qquad (\nu > n).$$

Proof. The statement follows from the Fourier repeated-integral formula, since, for $k \geqslant 0$, $\alpha > 1$, $\beta \geqslant 0$,

$$T_k(n,\nu) = \nu^k\left(1-\frac{\nu}{n}\right)^{\alpha-1} \log^{-\beta}\left(\frac{C}{1-\nu/n}\right)$$

is of bounded variation and continuous for $0 \leqslant \nu < n$, where n is fixed, and converges to zero as $\nu \to n$.

3. We shall now consider the rth derived series of the Fourier series of a function $f(x)$, integrable-L, at a point where the rth generalized derivative exists. It follows from Lemma 2.2 that for fixed n, $\lambda^{(k)}_{\alpha,\beta}(n,t)$ is absolutely integrable and of bounded variation in $(0,\infty)$, when $k \geqslant 1$, $\alpha = k+1$, $\beta \geqslant 0$. Hence, by a theorem due to W. H. Young† and Lemma 2.3, we obtain

$$\int_0^\infty [f(x+t)+f(x-t)]\lambda^{(2r)}_{\alpha,\beta}(n,t)\, dt$$

$$= i^{2r} \sum_{\nu<n} \left[B\left(1-\frac{\nu}{n}\right)^{\alpha-1} \log^{-\beta}\left(\frac{C}{1-\nu/n}\right)\nu^{2r}(a_\nu \cos \nu x + b_\nu \sin \nu x)\right], \quad (3.01)$$

† Hobson (**4**), 583.

where $r \geqslant 1$, $\alpha = 2r+1$, $\beta \geqslant 0$, and

$$-\int_0^\infty [f(x+t)-f(x-t)]\lambda_{\alpha,\beta}^{(2r+1)}(n,t)\,dt$$

$$= -i^{2r+2}\sum_{\nu<n}\left[B\left(1-\frac{\nu}{n}\right)^{\alpha-1}\log^{-\beta}\left(\frac{C}{1-\nu/n}\right)\nu^{2r+1}(-a_\nu \sin \nu x + b_\nu \cos \nu x)\right], \tag{3.02}$$

where $r \geqslant 0$, $\alpha = 2r+2$, $\beta \geqslant 0$. Thus we have the following lemma:

LEMMA 3.1. *The expression*

$$L_{n,r}(f,x) \equiv (-1)^r \int_0^\infty [f(x+t)+(-1)^r f(x-t)]\lambda_{r+1,\beta}^{(r)}(n,t)\,dt \tag{3.03}$$

is the [n]*th*† *mean of order* ($r, \beta \geqslant 0$) *of the* r*th derived series of the Fourier series corresponding to* $f(x)$.

THEOREM 3.1. *The* r*th derived series of the Fourier series of a function* $f(x)$, *integrable-L, is summable* (α,β), *where* $\alpha = r$, $\beta > 1$, *to the* r*th generalized derivative whenever it exists.*

Proof. Investigate first the derived series of even order. Then (3.03) becomes

$$L_{n,2r}(f,x) = \int_0^\infty [f(x+t)+f(x-t)]\lambda_{2r+1,\beta}^{(2r)}(n,t)\,dt. \tag{3.04}$$

Consider only points where $f(x)$ possesses a generalized derivative of order $2r$, and thus (1.06) is satisfied. Choose ϵ positive, then h positive (but fixed) so that

$$|\omega_1(x,t)| < \epsilon \qquad (0 \leqslant t \leqslant h), \tag{3.05}$$

and then n so that $nh > e$. By Lemma 2.2

$$L_1 \equiv \int_h^\infty [f(x+t)+f(x-t)]\lambda_{2r+1,\beta}^{(2r)}(n,t)\,dt$$

$$= O\left(\log^{-\beta} nh \int_h^\infty \left|\frac{f(x+t)+f(x-t)}{t^{2r+1}}\right| dt\right),$$

thus

$$\lim_{n\to\infty} L_1 = 0 \qquad (r \geqslant 1, \beta > 0). \tag{3.06}$$

Next let us investigate

$$L_2 \equiv \int_0^h [f(x+t)+f(x-t)]\lambda_{2r+1,\beta}^{(2r)}(n,t)\,dt.$$

† Where [n] is the largest integer $\leqslant n$.

By means of (1.06) set

$$L_2 \equiv L_3 + L_4 + L_5, \tag{3.07}$$

where

$$\left.\begin{aligned} L_3 &\equiv \sum_{\nu=0}^{r-1} \frac{2f^{(2\nu)}(x)}{(2\nu)!} \int_0^h t^{2\nu} \lambda_{2r+1,\beta}^{(2r)}(n,t)\, dt \\ L_4 &\equiv \frac{2f^{(2r)}(x)}{(2r)!} \int_0^h t^{2r} \lambda_{2r+1,\beta}^{(2r)}(n,t)\, dt \\ L_5 &\equiv \frac{2}{(2r)!} \int_0^h \omega_1(x,t) t^{2r} \lambda_{2r+1,\beta}^{(2r)}(n,t)\, dt \end{aligned}\right\}. \tag{3.08}$$

After integrating by parts each of the $r-1$ terms of L_3 and observing that the integrated terms vanish as $n \to \infty$, for $\beta \geqslant 0$ (Lemma 2.2), we have

$$\lim_{n\to\infty} L_3 = 0 \qquad (\beta \geqslant 0). \tag{3.09}$$

Consider L_4. After integrating by parts $2r$ times and observing that the integrated terms vanish at both limits as above, we see that the discussion of L_4 reduces to that of

$$L_6 \equiv \frac{2f^{(2r)}(x)}{1!} \int_0^h \lambda_{\alpha,\beta}(n,t)\, dt. \tag{3.10}$$

Replace the variable t by t/n and evaluate the integral by Lemma 2.3. Then for $\beta \geqslant 0$,

$$\lim_{n\to\infty} L_6 = f^{(2r)}(x). \tag{3.11}$$

It remains for us to investigate L_5. On account of (3.05)

$$|L_5| < K\epsilon \int_0^{nh} \left| \frac{t^{2r}}{n^{2r+1}} \lambda_{\alpha,\beta}^{(2r)}\left(n, \frac{t}{n}\right) \right| dt \qquad (\alpha = 2r+1).$$

Divide the interval $(0, nh)$ into $(0, l)$ and (l, nh) and investigate the two integrals independently.

$$L_7 \equiv \int_l^{nh} \left| \frac{t^{2r}}{n^{2r+1}} \lambda_{\alpha,\beta}^{(2r)}\left(n, \frac{t}{n}\right) \right| dt \leqslant K_1 \int_l^{nh} \frac{1}{t \log^{\beta} t}\, dt = O(1)$$

uniformly in n, when $\alpha = 2r+1$ and $\beta > 1$.

Finally

$$L_8 \equiv \int_0^l \left| \frac{t^{2r}}{n^{2r+1}} \lambda_{\alpha,\beta}^{(2r)}\left(n, \frac{t}{n}\right) \right| dt$$

is bounded in $(0,\infty)$, for all n and $\alpha \geqslant 1$, $\beta \geqslant 0$, since

$$\lambda^{(2r)}_{\alpha,\beta}(n,t/n) = i^{2r}n^{2r+1}\pi^{-1}Q_{2r,\alpha,\beta}(t).$$

We have now shown that for all values of n and for $\beta > 1$,

$$|L_5| < \epsilon K_2 < \epsilon_1, \tag{3.12}$$

where ϵ_1 can be made arbitrarily small with h. Thus, summing up the results of (3.06), (3.07), (3.09), (3.11), and (3.12), we see that for $\beta > 1$,

$$\lim_{n\to\infty} L_{n,2r}(f,x) = f^{(2r)}(x). \tag{3.13}$$

Combining (3.13) with Lemma 3.1 completes the proof for the case of the even derived series. The proof for the odd derived series is similar.

4. Throughout the remainder of this paper we shall consider $f(x)$ to be of bounded variation in $(0, 2\pi)$. Integrals of the Riemann-Stieltjes type will be employed.

We shall define the following functions:

$$\phi(t) = \tfrac{1}{2}[f(x+t)-f(x-t)-2tf'(x)] \qquad \psi(t) = [f(x+t)+f(x-t)-2f(x)] \tag{4.01}$$

$$\Phi(u) = \int_0^u |d\phi(t)| \qquad \Psi(u) = \int_0^u |d\psi(t)| \tag{4.02}$$

$$g_\beta(t) = Q_{0,1,\beta}(t) \qquad \bar{g}_\beta(t) = \overline{Q}_{0,1,\beta}(t) \tag{4.03}$$

$$A_\nu(x) = \nu(-a_\nu \sin \nu x + b_\nu \cos \nu x) \qquad B_\nu(x) = \nu(a_\nu \cos \nu x + b_\nu \sin \nu x) \tag{4.04}$$

LEMMA 4.1. *For $\beta > 0$, $g_\beta(t)$ and $\bar{g}_\beta(t)$ are continuous functions of t in the interval $(0,\infty)$, and, for $\beta > 1$, they are of bounded variation in the same interval.*

Proof. Continuity in $(0,\infty)$ follows from the fact that, for $\beta > 0$, both $\log^{-\beta}\{C/(1-u)\}\cos tu$ and $\log^{-\beta}\{C/(1-u)\}\sin tu$ are continuous functions of (t,u) in the region defined by

$$0 \leqslant u \leqslant 1, \qquad 0 \leqslant t < \infty, \tag{4.05}$$

and that $g_\beta(t)$ and $\bar{g}_\beta(t)$ converge to zero as $t \to \infty$. Bosanquet and Linfoot† have demonstrated that for $\beta > 1$, $g_\beta(t)$ is of bounded variation in $(0,\infty)$. The proof for the case of $\bar{g}_\beta(t)$ is similar.

† Bosanquet and Linfoot (1).

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LEMMA 4.2. *For $\beta > 0$*

$$\frac{2}{\pi}\int_0^\infty g_\beta(t)\cos tx\,dt = B\log^{-\beta}\left(\frac{C}{1-x}\right) \qquad (0 \leqslant x < 1),$$

$$= 0 \qquad (x \geqslant 1);$$

and

$$\frac{2}{\pi}\int_0^\infty \bar{g}_\beta(t)\sin tx\,dt = B\log^{-\beta}\left(\frac{C}{1-x}\right) \qquad (0 < x < 1),$$

$$= 0 \qquad (x = 0,\ x \geqslant 1).$$

Proof. The lemma follows from the Fourier repeated-integral formula since for $\beta > 0$, $B\log^{-\beta}\{C/(1-x)\}$ is of bounded variation and continuous in $0 \leqslant x < 1$, and tends to 0 as $x \to 1$.

5. We shall now consider the derived series of the Fourier series of a function $f(x)$ of bounded variation in $(0, 2\pi)$ at a point where

$$\text{(i)}\ f(x) \text{ and } f'(x) \text{ exist}, \tag{5.01}$$

$$\text{(ii)}\ \frac{1}{u}\int_0^u |d\phi(t)| = o(1), \text{ as } u \to 0. \tag{5.02}$$

It is known that these two conditions are satisfied almost everywhere.† We shall first demonstrate the following lemma:

LEMMA 5.1. *If $f(x)$ is of bounded variation, and $f(x)$ and $f'(x)$ exist at the point x, then for $\beta > 1$,*

$$\sum_{\nu<n} B\log^{-\beta}\left(\frac{C}{1-\nu/n}\right)A_\nu(x) - f'(x) = \lim_{L\to\infty}\frac{2}{\pi}\int_0^L g_\beta(nt)n\,d\phi(t), \tag{5.03}$$

where $A_\nu(x)$ is the general term of the derived series and the integral is taken in the Riemann-Stieltjes sense.

Proof. First we shall show that for $\beta > 1$,

$$\lim_{L\to\infty}\int_0^L g_\beta(nt)\,d[\tfrac{1}{2}\{f(x+t)-f(x-t)\}]$$

$$= -\lim_{L\to\infty}\int_0^L \tfrac{1}{2}[f(x+t)-f(x-t)]\,dg_\beta(nt). \tag{5.04}$$

† Consult Plessner (6) in the case of (5.02).

Now

$$\int_0^L g_\beta(nt)\, d[\tfrac{1}{2}\{f(x+t)-f(x-t)\}] = [g_\beta(nt)\tfrac{1}{2}\{f(x+t)-f(x-t)\}]_0^L - \\ - \int_0^L \tfrac{1}{2}[f(x+t)-f(x-t)]\, dg_\beta(nt). \quad (5.05)$$

Pass to the limit as $L \to \infty$. Because of Lemma 4.1, the integral

$$\int_0^\infty \tfrac{1}{2}[f(x+t)-f(x-t)]\, dg_\beta(nt)$$

exists† for $\beta > 1$. The integrated term of (5.05) vanishes at the lower limit with $[f(x+t)-f(x-t)]$, and at the upper limit with $g_\beta(nt)$, since it is $O((nt)^{-1}\log^{-\beta} nt)$ by Lemma 2.1. Thus the second integral of (5.04) exists and the relation (5.04) is satisfied. Similarly, using the fact that $\sin \nu t$ is an absolutely continuous function of t in $(0, L)$, we obtain

$$\lim_{L\to\infty} \int_0^L g_\beta(nt) A_\nu(x) \cos \nu t\, dt = -\lim_{L\to\infty} \int_0^L A_\nu(x) \frac{\sin \nu t}{\nu}\, dg_\beta(nt). \quad (5.06)$$

Next

$$\frac{2}{\pi}\int_0^L g_\beta(nt) n\, d\phi(t) = \frac{2}{\pi} n \int_0^L g_\beta(nt)\, d[\tfrac{1}{2}\{f(x+t)-f(x-t)\}] - \\ - \frac{2}{\pi} n f'(x) \int_0^L g_\beta(nt)\, dt. \quad (5.07)$$

Pass to the limit as $L \to \infty$. By Lemma 4.2 the last term becomes $-f'(x)$. Moreover, as $L \to \infty$, the first term on the right of (5.07) exists for $\beta > 1$, and thus under this assumption

$$\frac{2}{\pi}\int_0^\infty g_\beta(nt) n\, d\phi(t) = \frac{2}{\pi} n \int_0^\infty g_\beta(nt)\, d[\tfrac{1}{2}\{f(x+t)-f(x-t)\}] - f'(x). \quad (5.08)$$

Now since $f(x)$ is of bounded variation in $(0, 2\pi)$,

$$\sum_{\nu=1}^\infty \frac{A_\nu(x) \sin \nu t}{\nu}$$

converges boundedly to $\frac{1}{2}[f(x+t)-f(x-t)]$. Combining (5.04) and

† Hobson (5), 545.

(5.06), integrating term by term,† and then substituting from Lemma 4.2, we obtain

$$\frac{2}{\pi}n\int_0^\infty g_\beta(nt)\,d[\tfrac{1}{2}\{f(x+t)-f(x-t)\}] = \sum_{\nu=1}^{\infty}\frac{2}{\pi}n\int_0^\infty g_\beta(nt)A_\nu(x)\cos\nu t\,dt$$

$$= \sum_{\nu<n} B\log^{-\beta}\left(\frac{C}{1-\nu/n}\right)A_\nu(x). \quad (5.09)$$

From (5.08) and (5.09) the lemma follows.

THEOREM 5.1. *The derived series of the Fourier series of a function $f(x)$ of bounded variation is summable $(0,\beta)$ almost everywhere, for $\beta > 1$, to $f'(x)$.*

Proof. Assume that at the point x the two conditions (5.01) and (5.02) are satisfied. Choose ϵ arbitrarily small, then A so that

$$\frac{1}{u}\int_0^u |d\phi(t)| < \epsilon \qquad (0 \leqslant u \leqslant A). \quad (5.10)$$

Next choose n so that $nA > e$, and define I_1, I_2, and I_3 as follows:

$$\left.\begin{aligned} I_1 &\equiv \frac{2}{\pi}\int_0^{e/n} g_\beta(nt)n\,d\phi(t) \\ I_2 &\equiv \frac{2}{\pi}\int_{e/n}^{A} g_\beta(nt)n\,d\phi(t) \\ I_3 &\equiv \frac{2}{\pi}\int_{A}^{L} g_\beta(nt)n\,d\phi(t) \end{aligned}\right\}. \quad (5.11)$$

Since $g_\beta(nt)$ is bounded (Lemma 2.1), then by (5.02)

$$\lim_{n\to\infty} I_1 = o(1). \quad (5.12)$$

Next let us investigate I_2.

$$|I_2| \leqslant K_2\int_{e/n}^{A}\frac{1}{t\log^\beta nt}\,|d\phi(t)|.$$

Integrating by parts and observing that the integrated term vanishes for $\beta > 1$ as $n \to \infty$, we see that the discussion of I_2 reduces to that

† Young (7).

of the following expression:

$$\left|-K_2\int_{e/n}^{A}\Phi(t)t^{-2}\log^{-\beta}nt\,dt-\beta K_2\int_{e/n}^{A}\Phi(t)t^{-2}\log^{-\beta-1}nt\,dt\right|. \quad (5.13)$$

In the case of the first integral of (5.13) we may write, by (5.10),

$$\left|-K_2\int_{e/n}^{A}\Phi(t)t^{-2}\log^{-\beta}nt\,dt\right|<K_2\epsilon\int_{e/n}^{A}t^{-1}\log^{-\beta}nt\,dt$$

and for $\beta>1$, this term is arbitrarily small with ϵ, since $nA>e$. Also

$$\left|-\beta K_2\int_{e/n}^{A}\Phi(t)t^{-2}\log^{-\beta-1}nt\,dt\right|<K_2\beta\epsilon\int_{e/n}^{A}t^{-1}\log^{-\beta-1}nt\,dt$$

and similarly, for $\beta>1$, this term is arbitrarily small with ϵ.

Finally consider I_3.

$$|I_3|\leqslant K_3\int_{A}^{L}t^{-1}\log^{-\beta}nt\,|d\phi(t)|.$$

On account of the periodicity of $f(x)$, if q and s be chosen so that

$$2(q-1)\pi\leqslant A<2q\pi\leqslant 2(s-1)\pi<L\leqslant 2s\pi,$$

then $$|I_3|\leqslant K_3\left[\int_0^{2\pi}|d\phi(t)|\left\{\frac{1}{A\log^{\beta}nA}+\frac{1}{2\pi}\sum_{\nu=q}^{s-1}\frac{1}{\nu\log^{\beta}2n\pi\nu}\right\}\right].$$

The series $\displaystyle\sum_{\nu=q}^{\infty}\frac{1}{\nu\log^{\beta}2n\pi\nu}$ converges, for $\beta>1$, to a sum which tends to zero as $n\to\infty$. Thus for $\beta>1$,

$$\lim_{L\to\infty}\frac{2}{\pi}\int_0^{L}g_{\beta}(nt)n\,d\phi(t)\to 0,\text{ as }n\to\infty. \quad (5.14)$$

In view of the fact that the conditions (5.01) and (5.02) are satisfied almost everywhere, the theorem follows from (5.14) and Lemma 5.1.

6. We shall next investigate the summability of the series conjugate to the derived series of the Fourier series of a function $f(x)$, of bounded variation in $(0,2\pi)$, at a point where the following conditions are satisfied:

$$\text{(i)}\ \frac{1}{u}\int_0^{u}|d\psi(t)|\to 0,\quad\text{as}\quad u\to 0, \quad (6.01)$$

$$\text{(ii)}\ \lim_{\eta\to 0} -\frac{1}{\pi}\int_{\eta}^{\infty}\frac{\psi(t)}{t^2}\,dt \quad \text{exists.} \tag{6.02}$$

We state first the following relations:

$$\lim_{L\to\infty}\int_0^L \bar{g}_\beta(nt)\,d[\tfrac{1}{2}\{f(x+t)+f(x-t)\}]$$

$$= -\lim_{L\to\infty}\int_0^L \tfrac{1}{2}[f(x+t)+f(x-t)]\,d\bar{g}_\beta(nt), \tag{6.03}$$

$$\lim_{L\to\infty}\int_0^L \bar{g}_\beta(nt)B_\nu(x)\sin\nu t\,dt = \lim_{L\to\infty}\int_0^L \frac{B_\nu(x)}{\nu}\cos\nu t\,d\bar{g}_\beta(nt). \tag{6.04}$$

The existence of the Riemann-Stieltjes integrals and the proofs of the relations (6.03) and (6.04) are similar to those given for (5.04) and (5.06).

LEMMA 6.1. *If $f(x)$ is a function of bounded variation, then for $\beta > 1$,*

$$\sum_{\nu<n} B\log^{-\beta}\left(\frac{C}{1-\nu/n}\right)B_\nu(x) = \lim_{L\to\infty} -\frac{n}{\pi}\int_0^L \bar{g}_\beta(nt)\,d\psi(t), \tag{6.05}$$

where $B_\nu(x)$ is the general term of the series conjugate to the derived series and the integral is taken in the Riemann-Stieltjes sense.

Proof. We have

$$2\int_0^\infty \bar{g}_\beta(nt)\,d[\tfrac{1}{2}\{f(x+t)+f(x-t)\}] = \int_0^\infty \bar{g}_\beta(nt)\,d\psi(t).$$

The remainder of the proof of the lemma is similar to that of Lemma 5.1.

THEOREM 6.1. *The series conjugate to the derived series is summable $(0,\beta)$, for $\beta > 1$, to*

$$T \equiv \lim_{\eta\to 0} -\frac{1}{\pi}\int_{\eta}^{\infty}\frac{\psi(t)}{t^2}\,dt,$$

whenever this limit exists, provided that (6.01) *is satisfied and $\psi(t) = o(t)$ as $t \to 0$.*

Proof. It is sufficient to show that under these conditions

$$J \equiv \lim_{L\to\infty} -\frac{n}{\pi}\int_0^L \bar{g}_\beta(nt)\,d\psi(t) \to T, \text{ as } n\to\infty. \tag{6.06}$$

Choose ϵ arbitrarily small, then A so that

$$\frac{1}{u}\int_0^u |d\psi(t)| < \epsilon \qquad (0 \leqslant u \leqslant A). \tag{6.07}$$

Next choose n so that $nA > e$, and define J_1, J_2, J_3 as follows:

$$\left.\begin{aligned} J_1 &\equiv -\frac{n}{\pi}\int_0^{e/n} \bar{g}_\beta(nt)\, d\psi(t) \\ J_2 &\equiv -\frac{n}{\pi}\int_{e/n}^{A} \bar{g}_\beta(nt)\, d\psi(t) \\ J_3 &\equiv -\frac{n}{\pi}\int_{A}^{L} \bar{g}_\beta(nt)\, d\psi(t) \end{aligned}\right\}. \tag{6.08}$$

In the interval $(e/n, L)$,

$$\bar{g}_\beta(nt) = (1/nt) + h_\beta(nt),$$

where by (4.03) and Lemma 2.1

$$h_\beta(nt) = O\{(nt)^{-1}\log^{-\beta} nt\}.$$

Set $\quad J_2 \equiv J_2^* + J_2^{**}$ and $J_3 \equiv J_3^* + J_3^{**}$,

where

$$\left.\begin{aligned} J_2^* &\equiv -\frac{n}{\pi}\int_{e/n}^{A} (nt)^{-1}\, d\psi(t) \\ J_2^{**} &\equiv -\frac{n}{\pi}\int_{e/n}^{A} h_\beta(nt)\, d\psi(t) \end{aligned}\right\}, \tag{6.09}$$

with similar definitions for J_3^* and J_3^{**}. As in Theorem 5.1, the integrals J_1, J_2^{**}, and J_3^{**} are $o(1)$ as $n \to \infty$, for $\beta > 1$. Thus for n sufficiently large and $\beta > 1$,

$$\lim_{L\to\infty} -\frac{n}{\pi}\int_0^L \bar{g}_\beta(nt)\, d\psi(t) = \lim_{L\to\infty} -\frac{1}{\pi}\int_{e/n}^{L} t^{-1}\, d\psi(t) + o(1), \tag{6.10}$$

where $$-\frac{1}{\pi}\int_{e/n}^{\infty} t^{-1}\, d\psi(t) = \frac{1}{\pi}\left(\frac{n}{e}\right)\psi\left(\frac{e}{n}\right) - \frac{1}{\pi}\int_{e/n}^{\infty} t^{-2}\psi(t)\, dt.$$

Let $n \to \infty$, then, since $\psi(t) = o(t)$ as $t \to 0$,

$$\lim_{n\to\infty} -\frac{1}{\pi}\int_{e/n}^{\infty} t^{-1}\, d\psi(t) = \lim_{\eta\to 0} -\frac{1}{\pi}\int_{\eta}^{\infty} t^{-1}\, d\psi(t) = \lim_{\eta\to 0} -\frac{1}{\pi}\int_{\eta}^{\infty} t^{-2}\psi(t)\, dt$$

provided the limits exist. Thus the theorem is proved.

Let us now consider the restrictions of Theorem 6.1. It is known that (6.01) is satisfied almost everywhere.† Moreover, since $\Psi(t)$ is the total variation of $\psi(t)$ in the interval $(0, t)$, it follows that $\psi(t) = o(t)$ as $t \to 0$ almost everywhere. Plessner† has established the existence of

$$\lim_{\eta \to 0} \int_{\eta}^{\pi} \frac{\psi(t)}{\sin^2(\frac{1}{2}t)}\, dt$$

except at a set of points x of measure zero. By writing

$$\frac{1}{4}\int_{\eta}^{\pi} \frac{\psi(t)}{\sin^2(\frac{1}{2}t)}\, dt = \frac{1}{4}\int_{\eta}^{\pi} \psi(t)\left[\frac{1}{\sin^2(\frac{1}{2}t)} - \frac{4}{t^2}\right] dt + \int_{\eta}^{\pi} \frac{\psi(t)}{t^2}\, dt,$$

the existence of $$\lim_{\eta \to 0} -\frac{1}{\pi}\int_{\eta}^{\infty} \frac{\psi(t)}{t^2}\, dt$$

almost everywhere is demonstrated. Hence we have the following theorem.

THEOREM 6.2. *The series conjugate to the derived series of the Fourier series of a function of bounded variation is summable* $(0, \beta)$ *for* $\beta > 1$, *to*

$$T \equiv \lim_{\eta \to 0} -\frac{1}{\pi}\int_{\eta}^{\infty} \frac{\psi(t)}{t^2}\, dt$$

almost everywhere, where $\psi(t)$ *is defined by* (4.01).

† Plessner (6).

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THE FORMAL REDUCTION OF SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

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Introduction

In his treatment of the classical problem of systems of ordinary linear differential equations with constant coefficients, Jordan gives a remarkable elementary process* for exhibiting the invariant factors of the system and reducing it to an equivalent diagonal system of special form. So far as I know, it does not appear to have been noticed that the same method is applicable with slight modifications to systems with variable coefficients.

Consider the system

$$u_i \equiv \sum_{j=1}^{n} F_{ij} y_j = 0 \qquad (i = 1, 2, \ldots, n), \tag{1}$$

where (F_{ij}) are operators of the type

$$F \equiv a_0(x)\frac{d^m}{dx^m} + a_1(x)\frac{d^{m-1}}{dx^{m-1}} + \ldots + a_m(x), \tag{2}$$

of maximum order m, say. We shall show that (1) is equivalent to a diagonal system

$$H_1 w_1 = 0, \qquad H_2 w_2 = 0, \quad \ldots, \quad H_n w_n = 0, \tag{3}$$

where some of the operators (H_s) may reduce to unity. Further, if the equations (3) are arranged in the correct order, they have the property (analogous to that of invariant factors) that every *solution* of $H_k w = 0$ is a solution of $H_{k+1} w = 0$, and every *integrating factor* of $H_k w = 0$ is an integrating factor of $H_{k+1} w = 0$.

Inner and outer factors of operators

If two equations $(F_1 y = 0,\ F_2 y = 0)$ have common solutions, we may perform Brassine's process† (analogous to the extraction of the highest common factor of two polynomials) and find $(F_1 y \equiv F_1' Gy,\ F_2 y \equiv F_2' Gy)$, where $Gy = 0$ is the equation satisfied by the common

* Jordan, *Cours d'analyse*, iii. 174–9.

† See, for instance, Schlesinger, *Theorie der linearen Differentialgleichungen*, i. 42–5, or Picard, *Traité d'analyse*, iii. 554 (ed. 1908).

solutions. We may say that the operator G is the highest *inner common factor* of (F_1, F_2). Suppose now that we require the integrating factors common to the two equations; the classical solution is to construct the two adjoint equations and apply Brassine's process to them. We may, however, proceed more directly, if we observe that any common integrating factor of $(F_1 y = 0,\ F_2 y = 0)$ is also one of $F_1 P_1 y + F_2 P_2 y = 0$, where (P_1, P_2) are any operators. By an obvious counterpart of Brassine's process, in which the order of products of operators is reversed, we reduce the equations to the form $(HF_1'' = 0,\ HF_2'' = 0)$, where $Hy = 0$ admits every integrating factor common to the given equations. We shall then call H the highest *outer common factor* of the operators (F_1, F_2).

Reduction by rows and by columns

The passage between the equivalent systems (1), (3) is effected by a sequence of elementary steps of two kinds. In the first place, a pair of equations $(u_i = 0,\ u_j = 0)$ may be replaced by $(u_i + \lambda u_j = 0,\ u_j = 0)$, where λ is any linear operator; for either pair is a consequence of the other. If we perform a sequence of these steps and keep a record of what we have done, we obtain two associated operator matrices connecting $(u_i = 0)$ and $(v_i = 0)$, such that

$$u_i \equiv \sum_{j=1}^{n} A_{ij} v_j \qquad v_i \equiv \sum_{j=1}^{n} A'_{ij} u_j. \tag{4}$$

By substituting one set of formulae in the other, we see that the operators satisfy identities

$$\sum_{p=1}^{n} A_{ip} A'_{pj} = \delta_{ij} \qquad \sum_{p=1}^{n} A'_{ip} A_{pj} = \delta_{ij}, \tag{5}$$

where $\delta_{ij} = 0\ (i \neq j)$ or $\delta_{ij} = 1\ (i = j)$. This process we call *reduction by rows* of the given system.

But we can also make changes of variables, using instead of the pair (y_i, y_j) the pair $(y_i + \lambda y_j, y_j)$, where λ is any linear operator; we see that either pair can be explicitly written in terms of the other. The sets of variables (y_i), (w_i) are connected by a sequence of such steps, and, on eliminating intermediate terms, we find again two associated matrices of operators (B_{ij}), (B'_{ij}) such that we have, symbolically, $(y = Bw,\ w = B'y)$, as a shorthand for relations of the type (4) and (5).

Generalized Jordan process

First Stage.

For simplicity of exposition, we divide the reduction into two stages; we obtain first an equivalent system

$$G_1 z_1 = 0, \qquad G_2 z_2 = 0, \quad \ldots, \quad G_n z_n = 0, \tag{6}$$

which does not in general satisfy the supplementary conditions imposed on (3); it is then sufficient to consider the equations (6) pair by pair and show how they may be further reduced to the required form.

We shall suppose that a system has been 'simplified' if it has been shown to depend on one with fewer equations; it does not follow that the 'simpler' system is necessarily easier to solve. Further, of two systems having the same number of equations and variables, we define that one as the 'simpler' which has an operator (not identically zero) of lower order than any operator in the matrix of the other.

(i) If any operator of the matrix (F_{ij}) is of order zero, but does not vanish identically, we can express the accompanying variable explicitly in terms of the rest and eliminate it from the system; the new system has fewer equations than the old and is, by definition, simpler. For example, to fix the ideas, let F_{11} be unity; then y_1 is known in terms of $(y_2, y_3, \ldots, y_n)$. If we write

$$w_1 \equiv y_1 + F_{12} y_2 + \ldots + F_{1n} y_n, \tag{7}$$

one of our new variables is w_1 and one of our new equations is $H_1 w_1 \equiv w_1 = 0$. In practice, we merely eliminate y_1, keeping a record of its expression in terms of the variables of the simplified system.

(ii) If no operator (F_{ij}) is of order zero, we select the operator of lowest order of the matrix, or one of the set of the same minimum order. To fix the ideas, let us again suppose that it is F_{11}. We now try to reduce the system by rows to a simpler equivalent one; to do this we examine whether F_{11} is an inner (or right-hand) factor of the other operators in the same column $F_{21}, F_{31}, \ldots, F_{n1}$. If it is not, suppose that F_{21} is *not* divisible on the right by F_{11}; we have $F_{21} \equiv F'_{21} F_{11} + R_{21}$, say, where R_{21} is an operator of lower order than F_{11}; we can now replace $u_2 = 0$ by $-F'_{21} u_1 + u_2 = 0$, leaving the other equations unchanged. The modified system is evidently equivalent to the original one, and, by definition, is simpler. We note that R_{21} is not necessarily the operator of lowest order in the new system, since the new equation may happen to contain a lower one.

If at any stage an operator of order zero appears, we reduce the system by eliminating a variable, as in (i) above.

(iii) If we do not see any possibility of reduction by rows, we attempt a reduction by columns; to do this, we examine whether our operator of lowest order, F_{11}, is an outer (or left-hand) factor of each operator in the same row, $u_1 = 0$. If it is not, suppose that F_{12} is *not* divisible on the left by F_{11}; we have $F_{12} \equiv F_{11} F''_{12} + S_{12}$, say, where S_{12} is of lower order than F_{11}; introducing a new set of variables $(y_1 + F''_{12} y_2, y_2, y_3, \ldots, y_n)$ equivalent to the old, the system is again simplified.

(iv) We continue to apply processes (i), (ii), (iii) until no further reduction appears possible. Suppose that the new system* still has n equations and n variables, and that the operator of lowest order is again F_{11} (which does not imply any loss of generality, as we have only to rearrange the order of our equations and variables). By hypothesis, F_{11} must now be an outer factor of every operator of the new equation, which we call $u_1 = 0$; this can be written accordingly in the form

$$u_1 \equiv F_{11} z_1 \equiv F_{11}[y_1 + F''_{12} y_2 + F''_{13} y_3 + \ldots + F''_{1n} y_n] = 0, \qquad (8)$$

and defines a new variable z_1, such that the set of variables $(z_1, y_2, \ldots, y_n)$ is equivalent to $(y_1, y_2, \ldots, y_n)$. We can now eliminate y_1.

But F_{11} must also be an inner factor of every operator in the first column; this column of the matrix is unaltered by the change of variables, and its operators are of the form $F'_{21} F_{11}, F'_{31} F_{11}, \ldots, F'_{n1} F_{11}$ We can now replace the equations $(u_i = 0)$ by the equivalent set

$$u_1 = 0, \qquad -F'_{21} u_1 + u_2 = 0,$$
$$-F'_{31} u_1 + u_3 = 0, \quad \ldots, \quad -F'_{n1} u_1 + u_n = 0. \qquad (9)$$

The variable z_1 now appears only in the first equation of the new system; we have now

$$G_1 z_1 \equiv F_{11} z_1 = 0, \qquad (10)$$

together with a system of $n-1$ equations in $y_2, y_3, \ldots, y_n$.

By applying the same methods to this residual system, we can gradually evolve the diagonal system (6).

Second Stage.

We may set apart any of the equations (6) which are of the form $z_i = 0$, no further reduction being necessary. Let us consider any

* For simplicity, the notation (1) is retained schematically for the system at this stage of the reduction.

other pair of equations, say,

$$G_1 z_1 = 0, \qquad G_2 z_2 = 0, \tag{11}$$

where G_1 is of lower order than G_2 (or of the same order). Unless the operators already possess the characteristic property of the operators H_i, namely that G_1 is both an inner and an outer factor of G_2, further simplification is possible.

In the exceptional case, we shall have two identities of the form

$$G_2 z \equiv G_2' G_1 z, \qquad G_2 z \equiv G_1 G_2'' z. \tag{12}$$

We observe that it does not necessarily follow that G_2 can be written in the form $G_1 \Gamma G_1$, as may be readily verified by considering operators with constant coefficients.

For suppose that G_1 is *not* an outer factor of G_2. We replace (11) by the system

$$G_1 z_1 + G_2 z_2 = 0, \qquad G_2 z_2 = 0. \tag{13}$$

If $G_2 \equiv G_1 G_2'' + S_2$, we take new variables $(z_1 + G_2'' z_2,\ z_2)$ and write

$$G_1(z_1 + G_2'' z_2) + S_2 z_2 = 0, \qquad G_2 z_2 = 0. \tag{14}$$

Since S_2 is not zero and is of lower order than G_1, the system is, by definition, simpler than before; and as it is not of diagonal form, we can resume the sequence of reductions by the standard methods. By considering the process for the extraction of the highest common factor, it can be shown that the sum of the orders of the equations in two equivalent diagonal systems is the same; this point may be left to the reader.

Again, suppose G_1 is *not* an inner factor of G_2. We replace the variables $(z_1,\ z_2)$ by $(z_1,\ z_2 - z_1)$, and the equations become

$$G_1 z_1 = 0, \qquad G_2 z_1 + G_2 z_2' = 0, \tag{15}$$

where $z_2' \equiv z_2 - z_1$. The system (15) is reducible by rows; for if $G_2 \equiv G_2' G_1 + R_2$, we operate with G_2' on $G_1 z_1 = 0$ and subtract from the second equation, obtaining the equivalent system

$$G_1 z_1 = 0, \qquad R_2 z_1 + G_2 z_2' = 0. \tag{16}$$

Since R_2 is not zero and is of lower order than G_1, so that (16) is, by definition, simpler than (15). The reduction of (16) to a diagonal form may be completed as before.

Unless $(G_1,\ G_2)$ fulfil the conditions (12), simplification is always possible, and in the new pair of equations, one will be of lower and one of higher order than either of the old pair. For example, if $(G_1 z = 0,\ G_2 z = 0)$ have no common solution, and are of order

(n_1, n_2), we can obtain an identity

$$K_1 G_1 z + K_2 G_2 z = z. \tag{17}$$

Also, from the equation of order n_1+n_2 satisfied by every solution of either $G_1 z = 0$ or $G_2 z = 0$, we have the identity

$$Lz \equiv P_1 G_1 z \equiv P_2 G_2 z. \tag{18}$$

The general solution of $Lz = 0$ is $z = z_1+z_2$, where $G_1 z_1 = 0$, $G_2 z_2 = 0$. But we can also express (z_1, z_2) in terms of z; for we have

$$z_1 = (K_1 G_1 + K_2 G_2) z_1 = K_2 G_2 z_1,$$

or

$$z_1 = K_2 G_2 (z_1+z_2) = K_2 G_2 z, \tag{19}$$

so that the solution of the system (11) is

$$z_1 = K_2 G_2 z, \qquad z_2 = K_1 G_1 z, \tag{20}$$

where z is any solution of the single equation $Lz = 0$. In the complete process, a second variable w would appear, such that (z_1, z_2) and (z, w) are equivalent, and we should have the equation $w = 0$, forming with $Lz = 0$ the reduced system.

Thus any pair of the equations (6) can be replaced by an equivalent pair, having the special property (12) that one is both an inner and an outer factor of the other. This is the characteristic property of the operators (H_i) of (3). So long as any two operators of the system (6) do not possess this property, a further reduction is possible. In a diagonal system of the type (6) taken at random, it is improbable that the operators will have common inner or outer factors, and we shall usually find that all the variables z can be made to depend on a single variable w.

THE MOTION OF THE PROBABILISTIC FLUID OF PRESENCE OF THE ELECTRON IN RELATIVISTIC WAVE MECHANICS

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THE formula obtained by me* for the probability of the presence of the electron, with regard to the relativistic wave mechanics prior to Dirac's modifications, allows us to go farther in the study of the behaviour of the same probability.

The following note derives an equation for the motion of an electron in an electromagnetic field from a relativistic wave-equation (with regard to the new meaning of the *motion of an electron*). The new equation contains certain terms not appearing in the corresponding classical equation.

Having written (under a general form) the de Broglie-Schrödinger function

$$\psi = ae^{2\pi i\phi/h} \qquad (h = \text{Planck's constant}), \tag{1}$$

where a and ϕ are both real functions of the space-time coordinates, and having substituted (1) in the fundamental relativistic equation,† which governs the above function, we obtain the two following equations:

$$\left.\begin{aligned} &a\,\Box\,\phi - 2(\mathbf{U} - \operatorname{Grad}\phi)\times\operatorname{Grad} a = 0 \\ &-\frac{h^2}{4\pi^2 a}\,\Box\, a + (\mathbf{U} - \operatorname{Grad}\phi)^2 + m^2c^2 = 0 \end{aligned}\right\}, \tag{2}$$

where‡

$$\Box \equiv \sum_{s=1}^{4}\frac{\partial^2}{\partial y_s^2} = \Delta_2 + \frac{\partial^2}{\partial y_4^2}$$

with $\qquad y_1 = x, \qquad y_2 = y, \qquad y_3 = z, \qquad y_4 = ict,$

and where we have introduced the two four-dimensional vectors

$$\mathbf{U} \equiv \left(\frac{\epsilon}{c}A_x, \frac{\epsilon}{c}A_y, \frac{\epsilon}{c}A_z, \frac{i\epsilon}{c}V\right)$$

$$\operatorname{Grad}\phi \equiv \left(\frac{\partial\phi}{\partial y_1}, \frac{\partial\phi}{\partial y_2}, \frac{\partial\phi}{\partial y_3}, \frac{\partial\phi}{\partial y_4}\right);$$

* Presented at the International Congress of the Mathematicians in Zurich (September 1932). See *Verhandlungen*, ii. 309.

† Cf. Gordon, *Zeits. für Phys.* 40 (1927), 119, equation (9).

‡ We use (following the Italian convention) the symbol $\times$ as a sign of scalar multiplication, and the symbol $\wedge$ as a sign of vector multiplication.

the constants $-\epsilon$ and m denote respectively the charge and mass, and finally V and $\mathbf{A}$ denote the scalar potential and vector potential relative to an assigned Maxwellian field, in which our electron is supposed to be susceptible of free presence.

The first of (2), by means of Gordon's four-dimensional vector* and of the classical circumstance

$$\frac{1}{c}\frac{\partial V}{\partial t}+\sum_{x,y,z}\frac{\partial A_x}{\partial x}=0,$$

leads to the continuity equation

$$\frac{\partial \rho}{\partial t}+\operatorname{div}(\rho \mathbf{w})=0,$$

where ρ represents for us the density of the probabilistic fluid of presence of the electron, while $\mathbf{w}$ denotes the (vectorial) velocity of the fluid itself.

Finally, introducing the quantity

$$Q=\frac{h^2}{8\pi^2c^2ma}\left(\frac{\partial^2 a}{\partial t^2}-c^2\Delta_2 a\right)$$

and recalling the second of (2) together with Gordon's four-dimensional vector, we obtain our general (*exact*) formula, mentioned by us at the beginning of this paper,

$$\rho=\left(1+\frac{2Q}{mc^2}\right)^{\frac{1}{2}}\left(1-\frac{w^2}{c^2}\right)^{-\frac{1}{2}}.$$

Let us not forget to notice that substituting ka in the place of a (with $k=$ constant different from zero) the equations (2) remain unaltered; and so we may substitute ka for a in this formula.

It may be useful to remember that, after having written the components of Gordon's $\mathbf{J}$ four-dimensional vector in the form

$$J_1=m\rho w_x, \qquad J_2=m\rho w_y, \qquad J_3=m\rho w_z, \qquad J_4=icm\rho,$$

we obtain† $$m\rho\mathbf{w}=a^2\left(\frac{\epsilon}{c}\mathbf{A}-\operatorname{grad}\phi\right). \tag{3}$$

Let us write, for brevity,

$$\mu=m\left(1+\frac{2Q}{mc^2}\right)^{\frac{1}{2}}\left(1-\frac{w^2}{c^2}\right)^{-\frac{1}{2}}$$

* Loc. cit., p. 121, equation (14).

† See also L. de Broglie, 'The wave mechanics and the atomic structure of matter and of radiation', in *Selected Papers on Wave Mechanics* (1929), by L. de Broglie and L. Brillouin, p. 134.

and keep in mind (3); we shall have

$$\mu\mathbf{w} = \frac{\epsilon}{c}\mathbf{A} - \operatorname{grad}\phi, \tag{4}$$

the meaning of the symbol grad (written with small initial) being evident; and then

$$\frac{d}{dt}(\mu\mathbf{w}) = \frac{\epsilon}{c}\frac{\partial \mathbf{A}}{\partial t} - \operatorname{grad}\left(\frac{\partial\phi}{\partial t}\right) + (\mathbf{w}\times\operatorname{grad})(\mu\mathbf{w}), \tag{5}$$

indicating by $(\mathbf{w}\times\operatorname{grad})(\mu\mathbf{w})$ the vector, which has

$$\mathbf{w}\times\operatorname{grad}(\mu w_x), \qquad \mathbf{w}\times\operatorname{grad}(\mu w_y), \qquad \mathbf{w}\times\operatorname{grad}(\mu w_z)$$

for its components.

Let us notice also that the second of (2) leads to

$$\left(\epsilon V + \frac{\partial\phi}{\partial t}\right)^2 = m^2c^4 + (\epsilon\mathbf{A} - c\operatorname{grad}\phi)^2 + 2mc^2Q. \tag{6}$$

From which, taking the gradient, we have

$$2\left(\epsilon V + \frac{\partial\phi}{\partial t}\right)\operatorname{grad}\left(\epsilon V + \frac{\partial\phi}{\partial t}\right) = c^2\operatorname{grad}(\mu\mathbf{w}\times\mu\mathbf{w}) + 2mc^2\operatorname{grad}Q.$$

But

$$\begin{aligned}\operatorname{grad}(\mu\mathbf{w}\times\mu\mathbf{w}) &= 2(\mu\mathbf{w}\times\operatorname{grad})(\mu\mathbf{w}) + 2\mu\mathbf{w}\wedge\operatorname{curl}(\mu\mathbf{w}) \\ &= 2\mu(\mathbf{w}\times\operatorname{grad})(\mu\mathbf{w}) + 2\frac{\epsilon}{c}\mu\mathbf{w}\wedge\operatorname{curl}\mathbf{A};\end{aligned}$$

so that

$$\begin{aligned}&\left(\epsilon V + \frac{\partial\phi}{\partial t}\right)\operatorname{grad}\left(\epsilon V + \frac{\partial\phi}{\partial t}\right) \\ &\qquad = c^2\mu(\mathbf{w}\times\operatorname{grad})(\mu\mathbf{w}) + c\epsilon\mu\mathbf{w}\wedge\operatorname{curl}\mathbf{A} + mc^2\operatorname{grad}Q. \end{aligned}\tag{7}$$

From (6) together with (4), we have also

$$\epsilon V + \frac{\partial\phi}{\partial t} = c^2m\sqrt{\left\{1 + \left(\frac{\mu}{m}\right)^2\left(\frac{w}{c}\right)^2 + \frac{2Q}{mc^2}\right\}}, \tag{8}$$

where the radical is assumed by us to have its arithmetical value.

Now, *introducing the hypothesis of the smallness of the ratios* w^2/c^2 and $2Q/mc^2$, we shall be able to substitute, in (8), for the second member the expression

$$c^2m\sqrt{\left\{1 + \left(\frac{w}{c}\right)^2 + \frac{2Q}{mc^2}\right\}}$$

and then replace this by μc^2.

Therefore, instead of (7), we shall write

$$\operatorname{grad}\left(\frac{\partial\phi}{\partial t}\right) = -\epsilon \operatorname{grad} V + (\mathbf{w}\times\operatorname{grad})(\mu\mathbf{w}) + \frac{\epsilon}{c}\mathbf{w}\wedge\operatorname{curl}\mathbf{A} + \frac{m}{\mu}\operatorname{grad} Q.$$

Finally, substituting in (5), we obtain

$$\frac{d}{dt}(\mu\mathbf{w}) = \epsilon\left(\operatorname{grad} V + \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}\right) - \frac{\epsilon}{c}\mathbf{w}\wedge\operatorname{curl}\mathbf{A} - \frac{m}{\mu}\operatorname{grad} Q$$

or

$$\frac{d}{dt}(\mu\mathbf{w}) = -\epsilon\left(\mathbf{E} + \frac{1}{c}\mathbf{w}\wedge\mathbf{H}\right) - \frac{m}{\mu}\operatorname{grad} Q, \tag{9}$$

with evident meaning of the symbols. This is the desired equation.

As we can see (let us say in passing) the corresponding classical equation

$$\frac{d}{dt}\left(m\mathbf{w}\left(1-\frac{w^2}{c^2}\right)^{-\frac{1}{2}}\right) = -\epsilon\left(\mathbf{E} + \frac{1}{c}\mathbf{w}\wedge\mathbf{H}\right)$$

which differs from (9), might be obtained merely by taking no account of $(m/\mu)\operatorname{grad} Q$ in the second member of (9) and by replacing in the first member the quantity μ by

$$m\left(1-\frac{w^2}{c^2}\right)^{-\frac{1}{2}}.$$

ON SOME CONGRUENCES ASSOCIATED WITH A SUB-VARIETY IN A V_n

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1. Introduction†

It is well known that the notions of curvatures and normals of a curve can be immediately generalized to relate to a congruence of vectors defined along a curve. Thus, if ζ^α is a congruence of unit vectors along a curve C in a Riemannian space V_n, we can define $n-1$ unit congruences, $\underset{2}{\zeta^\alpha}, \underset{3}{\zeta^\alpha},\ldots, \underset{n}{\zeta^\alpha}$, and $n-1$ scalar functions $\underset{1}{k},\ldots, \underset{n-1}{k}$, as follows:‡ $\underset{r\,r+1}{k}\,\underset{r+1}{\zeta^\alpha}$ is the component of $D\underset{r}{\zeta^\alpha}/Ds$ perpendicular to the r-space determined by the vectors $\underset{1}{\zeta^\alpha},\ldots, \underset{r}{\zeta^\alpha}$, where $\underset{1}{\zeta^\alpha} \equiv \zeta^\alpha$ and D/Ds denotes absolute differentiation along C, and the k's are all positive. The congruences $\underset{1}{\zeta^\alpha},\ldots, \underset{n}{\zeta^\alpha}$ are mutually orthogonal and satisfy the equations

$$\frac{D\underset{r}{\zeta^\alpha}}{Ds} = \underset{r\,r+1}{k}\,\underset{r+1}{\zeta^\alpha} - \underset{r-1\,r}{k}\,\underset{r-1}{\zeta^\alpha} \qquad \left(r = 1,\ldots,n-1;\ \underset{0}{k} = \underset{n}{k} = 0\right). \tag{1}$$

(When ζ^α is tangential to the curve, these are the Frenet formulae and $\underset{r}{k} = \kappa_r$.)

We shall call the congruence $\underset{r+1}{\zeta^\alpha}$ the 'rth normal congruence of the congruence ζ^α' and $\underset{r}{k}$ the 'rth curvature§ of the congruence ζ^α'.

McConnell (**6**) shows how the curvatures may be expressed in terms of 'rates of turning' of the normals.

If C lies in a sub-space V_m of V_n, and if the congruence ζ^α is tangential to V_m, there are also defined, in a similar manner, $m-1$

† The notation in §§ 1, 2 is that of (**3**) and (**4**). A_n denotes an affine n-dimensional space, V_n a Riemannian space, E_n a Euclidean space.

‡ This simple and direct method of generating successive normals is given by McConnell (**6**).

§ Eisenhart (**2**) uses the terms 'associate directions of order r' and 'associate curvatures of order r' (extending Bianchi's terminology); Struik (**9**) calls $\underset{r}{k}$ the 'rth curvature of the curve C relative to the congruence ζ^α'. There appears to be no ambiguity in the terminology used in the text.

'normal congruences', $\underset{2}{\zeta'^i},\ldots,\underset{m}{\zeta'^i}$, and $m-1$ 'curvatures', $\underset{1}{k'},\ldots,\underset{m-1}{k'}$, *relative to* V_m, satisfying the relations

$$\frac{D\underset{r}{\zeta'^i}}{Ds} = \underset{r}{k'}\,\underset{r+1}{\zeta'^i} - \underset{r-1}{k}\,\underset{r-1}{\zeta'^i} \qquad \left(r = 1,\ldots,m-1;\ \underset{0}{k'} = \underset{m}{k'} = 0\right), \tag{2}$$

where $\underset{1}{\zeta'^i} \equiv \zeta'^i = \theta^i_{.\alpha}\zeta^\alpha$.

We shall define a congruence along a curve in V_m as being '*conjugate of order* p', if its first p normal congruences relative to V_n are all tangential to V_m; p is essentially not greater than $m-1$. This extension of the term 'conjugate' is justified in § 2.

An asymptotic line of order p is, therefore, the envelope of a conjugate congruence of order p, and the object of the earlier part of this paper is to obtain some properties of conjugate congruences corresponding to those of asymptotic lines given in (3). In particular, it is shown that, for a V_{n-1} in V_n, the Beltrami-Enneper formula is only a special case of a simple formula applicable to *any* curve in V_{n-1}.

In the later part (§§ 3, 4) we consider the more general case in which V_m (or the portion of it contiguous to the curve) is replaced by a non-holonomic variety V_1^m, an important feature of the generalization being that V_1^m is not restricted to be tangential to the curve.

2. Congruences tangential to a V_m

By arguments similar to those in (3), 25–7, it is easily shown† that each of the sets of conditions $[A;p]$, $[B;p]$, $[C;p]$ below is necessary and sufficient in order that the congruence ζ^α be conjugate of order p:

$$[A;p] \qquad \underset{r+1}{\zeta^\alpha} = \underset{r+1}{\zeta'^i}\theta'^\alpha_i \qquad (r = 1,\ldots,p;\ \alpha = 1,\ldots,n),$$

$$[B;p] \qquad \underset{r}{k} = \underset{r}{k'} \qquad (r = 1,\ldots,p);$$

i.e. the first p normal congruences and curvatures relative to V_n are identical with those relative to V_m;

$$[C;p] \qquad \Omega_{ij}^{\alpha}\xi'^i\underset{r}{\zeta'^j} = 0 \qquad (r = 1,\ldots,p;\ \alpha = 1,\ldots,n),$$

where ξ'^i is the tangent vector to the curve, regarded as a vector in V_m, and Ω_{ij}^{α} is the Eulerian curvature tensor of V_m in V_n.

For a V_{n-1} in V_n the conditions $[C;1]$ for a conjugate congruence

† The reader will find no difficulty in supplying the proofs.

of the first order reduce to the single condition

$$\omega_{ij}\xi'^i\zeta'^j = 0,$$

where ω_{ij} is the second fundamental tensor of V_{n-1}; so that the directions of ζ'^i and ξ'^i are *conjugate in the ordinary sense*. The notion of conjugate congruences of higher orders is therefore an extension of the ordinary notion of conjugate directions.

Before proceeding further, we consider whether there are any restrictions on the existence of conjugate congruences.†

The V_n-vectors $\Omega_{ij}^{\cdot\cdot\alpha}\xi'^i$, that is, $D\theta_j^{\ \alpha}/Ds$, for $j = 1,\ldots, m$, at a point of C, all lie in the first normal space‡ of V_m relative to V_n, and possibly in a space of lower dimensionality. Denoting the dimensionality of that space by M, let $\underset{\sigma}{\chi^\alpha}$ $(\sigma = 1,\ldots, M)$ be any M mutually orthogonal unit vectors in that space; then equations $[C;p]$ are equivalent to

$$\overset{\sigma}{\omega}_i\underset{r}{\zeta'^i} = 0 \qquad (r = 1,\ldots,p;\ \sigma = 1,\ldots, M), \tag{3}$$

where

$$\overset{a}{\omega}_i \equiv \overset{\sigma}{\omega}_{ji}\xi'^j \equiv -\Omega_{ji}^{\cdot\cdot\alpha}\underset{\sigma}{\chi_\alpha}\xi'^j = -\underset{\sigma}{\chi_\alpha}\frac{D\theta_i^{\ \alpha}}{Ds} = \theta_i^{\ \alpha}\frac{D\underset{\sigma}{\chi_\alpha}}{Ds}. \tag{4}$$

Equations (3) are equivalent to

$$\overset{\sigma}{\omega}_i\frac{D^r\zeta'^i}{Ds^r} = 0 \qquad (r = 0, 1,\ldots, p-1;\ \sigma = 1,\ldots, M),$$

which are easily seen to be equivalent to

$$\zeta'^i\frac{D^r\overset{\sigma}{\omega}_i}{Ds^r} = 0 \qquad (r = 0, 1,\ldots, p-1;\ \sigma = 1,\ldots, M),$$

that is, to

$$\zeta'^i\overset{\sigma}{\underset{r}{\omega}}_i = 0 \qquad (r = 1,\ldots, p;\ \sigma = 1,\ldots, M), \tag{5}$$

where $\overset{\sigma}{\underset{r}{\omega}}_i$ is the $(r-1)$th normal congruence of $\overset{\sigma}{\omega}_i$ relative to V_m.

The number, N (say), of *independent* equations (5) is equal to the number of independent vectors $\overset{\sigma}{\underset{r}{\omega}}_i$ $(r = 1,\ldots, p;\ \sigma = 1,\ldots, M)$ at the point considered; thus $N \leqslant m$ and also $N \leqslant pM \leqslant pm_1$, where m_1 is the dimensionality of the first normal space of V_m. For the existence of a conjugate congruence of order p along a given curve

† I am indebted to Mr. A. G. Walker for suggesting the consideration of existence-conditions.

‡ *Vide* (4), 45.

C, it is necessary and sufficient that $N < m$ at every point of C. Hence, if $m_1 < m/p$ at every point of C, there certainly exist conjugate congruences of order p along C.

In particular, if every point of C is an axial point of V_m (in which case $m_1 = 1$), there exist conjugate congruences of all orders along C. Further, since $m_1 \leqslant n-m$, we have the following theorems:

I. *Conjugate congruences of all orders less than $m/(n-m)$ exist along any curve in a V_m in V_n.*

II. *If $m > \frac{1}{2}n$, conjugate congruences of the first order exist along any curve in a V_m in V_n.*

III. *Conjugate congruences of all orders exist along any curve in a V_{n-1} in V_n.*

Also, since $M = 1$ in this last case, the conjugate congruence of highest order (viz. $n-2$) along a given curve is unique, in general.

The mth curvature relative to V_n of a conjugate congruence of highest order (viz. $m-1$), if such exist, is given by the relation†

$$\begin{aligned} k^2_m &= \Omega_{ij}{}^{\alpha}\Omega^{j}_{k\cdot\alpha}\xi'^i\xi'^k = \Omega_{ik}{}^{\alpha}\Omega^{j}_{j\cdot\alpha}\xi'^i\xi'^k - (m-1)\mathrm{H} \\ &= mN_{\alpha}\Xi^{\alpha} - (m-1)\mathrm{H}, \end{aligned}$$

where N^{α} is the mean normal curvature vector of V_m, Ξ^{α} is the normal curvature vector in the direction of the curve C, and H is the mean relative curvature of V_m in V_n in the direction of the curve. Hence

$$\boxed{k^2_m = mJ\nu\cos\theta - (m-1)\mathrm{H}}\ , \tag{6}$$

where J is the mean normal curvature of V_m, ν is its normal curvature in the direction of the curve, and θ is the angle between the vectors Ξ^{α} and N^{α}.

For $m = n-1$, the directions of the vectors Ξ^{α} and N^{α} coincide in that of the normal to V_{n-1}, and hence, for the conjugate congruence of order $n-2$ along any curve in V_{n-1},

$$k^2_{n-1} = (n-1)J\nu - (n-2)\mathrm{H}; \tag{7}$$

while, for a V_2 in E_3, the second curvature of the conjugate congruence (of the first order) along any curve is equal to

$$\sqrt{(2J\nu - K')}, \tag{8}$$

where K' is the Gaussian curvature of the surface V_2.

† Again cf. the proof in (**3**), § 7.

The first term on the right-hand side of (6) vanishes if

$$\Omega_{j\cdot\alpha}^{\cdot j}\Omega_{i\bar{k}}^{\cdot\alpha}\xi'^i\xi'^k = 0. \tag{9}$$

If J is zero at a point of V_m, every direction in V_m at that point satisfies this equation; thus, for example, the first term in (6) vanishes for every curve in a minimal sub-space of V_n.

If J is not zero, the directions ξ'^i satisfying (9) generate an $(m-1)$-dimensional quadratic cone at each point of V_m. This cone contains the asymptotic directions (when such exist†), corresponding to the case $\nu = 0$; the remaining directions, which exist only if $m < n-1$ and $m_1 > 1$, correspond to the case $\theta = \pm\frac{1}{2}\pi$.

When the curve C is an asymptotic line, the congruence ζ'^i is tangential to C, $\underset{m}{k}$ is equal to $\underset{m}{\kappa}$, and the formulae (6)–(8) reduce to the classical Beltrami-Enneper formula and its generalizations, (**3**), 30–1.

3. Congruences in a V_1^m in V_n.

If $\underset{i}{B^\alpha}$ $(i = 1,\ldots,m)$ are m $(< n)$ linearly independent vector fields defined at points of a sub-space V_l of V_n, the aggregate of the vector-spaces determined by them constitutes a non-holonomic variety V_l^m in V_n.

A connexion $\Gamma'^{\cdot i}_{jc}$, 'induced' in V_l^m by V_n, is given by‡

$$\Gamma'^{\cdot i}_{jc} = B^i_{\cdot\alpha}\left(\frac{\partial B_j^{\cdot\alpha}}{\partial y^c} + \Gamma^\alpha_{\lambda\mu}B_j^{\cdot\lambda}\theta_c^{\cdot\mu}\right), \tag{10}$$

where (i) y^c $(c = 1,\ldots,l)$ are coordinates in V_l, and $\theta_c^{\cdot\mu} \equiv \partial x^\mu/\partial y^c$; (ii) $B_i^{\cdot\alpha} = \overset{j}{e'_i}\underset{j}{B^\alpha}$, where $\overset{j}{e'_i}$ $(j = 1,\ldots,m)$ are the fundamental covariant vectors ('Massvektoren') in V_l^m corresponding to the system§|| $\underset{i}{B^\alpha}$, and $B^i_\alpha B_j^{\cdot\alpha} = \delta^i_j$; (iii) $\Gamma^\alpha_{\lambda\mu}$ are the coefficients of the connexion of V_n.

If ∇_c denotes covariant differentiation 'secondo R. Lagrange' (the type of differentiation, applied to tensors having several distinct sets

† The number of asymptotic directions is ∞^{m-m_1}, 2^{m_1}, or 0 according as $m_1 <, =, > m-1$ (Struik (**9**), p. 110).

‡ Schouten (**7**); Schouten und van Kampen (**8**) (for A_n^m in A_n); Bortolotti (**1**) (for A_m^p pseudonormal to A_m in A_n).

§ That is, $\overset{j}{e'_i}$ are the vectors whose components in the given system are $(1, 0,\ldots, 0)$; $(0, 1,\ldots, 0)$; ...; $(0,\ldots, 0, 1)$ [cf. (**1**), (**8**)].

|| So far as Latin indices i, j, k,... are concerned, contravariance and covariance is with respect to linear transformations of the vectors $\underset{i}{B^\alpha}$.

of indices, which Bortolotti employs to such advantage†),

$$\nabla_c B_i^{\ \alpha} = \frac{\partial B_i^{\ \alpha}}{\partial y^c} + \Gamma^{\alpha}_{\lambda\mu} B_i^{\ \lambda} \theta_c^{\ \mu} - \Gamma'^{j}_{ic} B_j^{\ \alpha}. \tag{11}$$

This may be called the Eulerian curvature tensor of V_l^m in V_n, and will be denoted by $\Omega_{ci}^{\ \ \alpha}$.

Since it is only the portion of V_m contiguous to the curve C which enters into the considerations of § 2, we shall restrict ourselves, without loss of generality, to the case of a V_1^m formed by m linearly independent vector fields $\underset{i}{B}{}^{\alpha}$ along a curve C in V_n. In this case there are m^2 coefficients of the induced connexion,

$$\Gamma'^{i}_{j1} = B^i_{\ .\alpha}\left(\frac{dB_j^{\ \alpha}}{ds} + \Gamma^{\alpha}_{\lambda\mu} B_j^{\ \lambda} \xi^{\mu}\right), \tag{12}$$

where ξ^{μ} is the unit tangent vector to the curve; and the Eulerian curvature tensor of V_1^m is

$$\Omega_{1i}^{\ \ \alpha} \equiv \frac{DB_i^{\alpha}}{Ds} = \frac{dB_i^{\ \alpha}}{ds} + \Gamma^{\alpha}_{\lambda\mu} B_i^{\ \lambda} \xi^{\mu} - \Gamma'^{j}_{i1} B_j^{\ \alpha}. \tag{13}$$

We shall write this as $\Omega_i^{\ \alpha}$.

If ζ^{α} is a congruence of unit vectors along C in V_1^m and ζ'^{i} are its components in V_1^m (viz. $B^i_{\ .\alpha}\zeta^{\alpha}$), we obtain two sets of normal congruences and two sets of curvatures—one set relative to V_n, the other relative to V_1^m—satisfying equations analogous to (1) and (2).

If we define a congruence in V_1^m as 'conjugate of order p' ($p < m$) if its first p normal congruences relative to V_n all lie in V_1^m, then, proceeding in a manner analogous to that in § 2 and remembering that

$$B^i_{\ .\alpha}\Omega_j^{\ \alpha} = 0, \tag{14}$$

as is easily verified from (12) and (13), we find the following necessary and sufficient conditions for a conjugate congruence of order p:

$$[A;p] \qquad \underset{r+1}{\zeta^{\alpha}} = \underset{r+1}{\zeta'^{i}} B_i^{\ \alpha} \qquad (r = 1,\ldots,p),$$

$$[B;p] \qquad \underset{r}{k} = \underset{r}{k'} \qquad (r = 1,\ldots,p),$$

$$[C;p] \qquad \Omega_i^{\ \alpha} \underset{r}{\zeta'^{i}} = 0 \qquad (r = 1,\ldots,p).$$

Also, in the case of a conjugate congruence of order $m-1$, if such exist,

$$\underset{m}{k^2} = \Omega_i^{\ \alpha}\Omega^i_{\ .\alpha}, \tag{15}$$

where

$$\Omega^i_{\ .\alpha} \equiv DB^i_{\ .\alpha}/Ds.$$

† It is fundamentally equivalent to the 'D-Symbolik' of van der Waerden and Schouten [*vide* e.g. (**8**)].

There are restrictions on the values of p analogous to those in § 2; while Theorems I–III of § 2 remain true if we replace the phrase 'along any curve in V_m' by 'in any V_1^m'.

If B_β^α is the fundamental tensor ('Einheitsaffinor') of V_1^m, viz. $B_\beta^\alpha = \sum_{i=1}^{m} B_i^\alpha B_{i\beta} = B_i{}^\alpha B^i{}_\beta$, and if we write $\Omega_\beta^\alpha \equiv DB_\beta^\alpha/Ds$, it is easily verified that $[C;p]$ and (15) take the alternative forms:†

$$[C;p]' \qquad \underset{r}{\Omega_\beta^\alpha \zeta^\beta} = 0 \qquad (r = 1,\ldots,p),$$

$$\underset{m}{k^2} = \tfrac{1}{2}\Omega_\beta^\alpha \Omega_\alpha^\beta. \qquad (15)'$$

If $m = n-1$, $\qquad B_\beta^\alpha = \delta_\beta^\alpha - \chi^\alpha \chi_\beta$,

where χ^α is the unit vector normal to V_1^{n-1}, and we find, for the conjugate congruence of order $n-2$,

$$\underset{n-1}{k^2} = \frac{D\chi^\alpha}{Ds}\frac{D\chi_\alpha}{Ds},$$

i.e. $\underset{n-1}{k}$ is equal to the first curvature of the congruence of normals to V_1^{n-1}.

In particular, for the V_1^{n-1} normal to the curve C, $\chi^\alpha = \xi^\alpha$ and $\underset{n-1}{k} = \underset{1}{\kappa}$.

If, when V_l^m is tangential‡ to V_l, we define a curve in V_l as an 'asymptotic line§ of order p of V_l^m' if its first p normals relative to V_n lie in V_l^m, the conditions for an asymptotic line of order p are, therefore,

$$\underset{r+1}{\xi^\alpha} = \underset{r+1}{\xi'^i} B_i{}^\alpha, \qquad \underset{r}{\kappa} = \underset{r}{\kappa'},$$

$$\Omega_{ci}{}^{\cdot\cdot\alpha} \underset{r}{\xi^{\prime\prime c}} \underset{r}{\xi'^i} = \Omega_{c\beta}{}^{\cdot\cdot\alpha} \underset{r}{\xi^{\prime\prime c}} \underset{r}{\xi^\beta} = 0,$$

where $\xi^{\prime\prime c}$ are the components of the tangent-vector in the coordinate

† To obtain these, we note that

$$\Omega_i{}^{\cdot\alpha} = B_\beta^\alpha \Omega_i{}^{\cdot\beta} + B_i{}^{\cdot\beta}\Omega_\beta^\alpha = B^{\cdot\beta}\Omega_{\cdot}^{\alpha},$$

whence $\qquad \underset{m}{k^2} = B_\beta^\alpha \Omega_\lambda^\beta \Omega_\alpha^\lambda.$

Also, since $\qquad B_\beta^\alpha B_\lambda^\beta = B_\lambda^\alpha,$

it follows that $\qquad B_\beta^\alpha \Omega_{\cdot\lambda}^{\beta} + B_{\cdot\lambda}^{\beta}\Omega_\beta^\alpha = \Omega_{\cdot\lambda}^{\alpha},$

and hence that $\qquad 2B_\beta^\alpha \Omega_\alpha^\lambda \Omega_\lambda^\beta = \Omega_{\cdot\lambda}^{\alpha}\Omega_{\cdot\alpha}^{\lambda}.$

‡ i.e. the tangent l-spaces to V_l are contained in V_l^m if $l \leqslant m$, or contain V_l^m if $l > m$.

§ Asymptotic lines in a non-holonomic variety (V_n^{n-1} in V_n) have been considered by Hlavatý (5).

system y^c (in V_l), $\underset{r}{\kappa'}$ is the rth curvature of the curve relative to V_l^m (*not* relative to V_l), and $\Omega_{cB}^{..\alpha} \equiv \nabla_c B_\beta^\alpha$.

For an asymptotic line of order $m-1$,

$$\underset{m}{\kappa^2} = \Omega_{ci}^{..\alpha}\Omega_{d.\alpha}^{..i}\xi^{rc}\xi^{rd} = \tfrac{1}{2}\Omega_{c\lambda}^{..\alpha}\Omega_{d\alpha}^{..\lambda}\xi^{rc}\xi^{rd}.$$

4. Congruences normal to a V_1^m in V_n $(m < n-1)$

There is an $(n-m)$-direction at each point of C perpendicular to V_1^m, and their aggregate constitutes a V_1^{n-m} 'complementary' to V_1^m. If B_β^α refers to V_1^m, the fundamental tensor of V_1^{n-m} is $C_\beta^\alpha \equiv \delta_\beta^\alpha - B_\beta^\alpha$, and the Eulerian curvature tensor of V_1^{n-m} is $(-\Omega_\beta^\alpha)$. Thus the conditions for a conjugate congruence of order p in V_1^{n-m} are

$$[C;p]' \qquad \Omega_\beta^\alpha \underset{r}{\zeta^\beta} = 0 \qquad (r = 1,...,p;\ \alpha = 1,...,n).$$

Since
$$\Omega_\beta^\alpha = \Omega_i^{..\alpha} B_{.\beta}^i + B_i^{..\alpha}\Omega_{.\beta}^i$$

and
$$B_{.\beta}^i \underset{r}{\zeta^\beta} = 0,$$

these conditions reduce (in number) to

$$[C;p] \qquad \Omega_{.\alpha}^{..i} \underset{r}{\zeta^\alpha} = 0 \qquad (r = 1,...,p;\ i = 1,...,m).$$

The case of $m = 1$ is of interest. If V_1^1 is composed of the congruence λ^α along the curve C, the conditions $[C;p]$ for a conjugate congruence of order p in the complementary V_1^{n-1} are

$$\frac{D\underset{1}{\lambda_\alpha}}{Ds}\underset{r}{\zeta^\alpha} = 0,$$

i.e.
$$\underset{1}{k}\underset{2}{\lambda_\alpha}\underset{r}{\zeta^\alpha} = 0 \qquad (r = 1,...,p), \tag{16}$$

where $\underset{2}{\lambda^\alpha}$ is the first normal congruence, and $\underset{1}{k}$ the first curvature, of the congruence λ^α.

If λ^α is a parallel congruence, every congruence in V_1^{n-1} is conjugate of order $n-2$; otherwise, conditions (16) are equivalent to

$$\underset{r}{\zeta^\alpha}\underset{\alpha}{\lambda_\alpha} = 0 \qquad (r = 2,...,p+1); \tag{17}$$

i.e. ζ^α must be perpendicular to the first p normal congruences of the given congruence.

The (unique) conjugate congruence of highest order is the congruence $\underset{n}{\lambda^\alpha}$.

If λ^α is taken to be tangential to C, then for ζ^α to be conjugate

of order p in the V_1^{n-1} normal to the curve, it is necessary and sufficient that ζ^α be normal to the osculating $(p+1)$-space of the curve at each point.

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AN EXTENSION OF THE METHOD OF GENERALIZED PLANE STRESS

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In this paper the method of generalized plane stress is extended to apply to plates of which the thickness decreases uniformly in one direction.

1. The stress function

We use cylindrical coordinates r, θ, z and take the planes $\theta = \pm\alpha$ to define the two sides of the plate.

Following a method analogous to that used in generalized plane stress,* let us assume that

$$\widehat{\theta\theta} = 0 \text{ throughout,}$$

$$\widehat{\theta z} = \widehat{\theta r} = 0 \text{ when } \theta = \pm\alpha.$$

Let

$$\breve{rr} = \frac{1}{2\alpha}\int_{-\alpha}^{+\alpha} \widehat{rr}\, d\theta, \qquad \breve{zz} = \frac{1}{2\alpha}\int_{-\alpha}^{+\alpha} \widehat{zz}\, d\theta, \qquad \breve{rz} = \frac{1}{2\alpha}\int_{-\alpha}^{+\alpha} \widehat{rz}\, d\theta. \quad (1.1)$$

The body-stress equations, in the case of equilibrium with no body force, are†

$$\left.\begin{aligned} &\frac{\partial \widehat{rr}}{\partial r} + \frac{1}{r}\frac{\partial \widehat{r\theta}}{\partial \theta} + \frac{\partial \widehat{rz}}{\partial z} + \frac{\widehat{rr} - \widehat{\theta\theta}}{r} = 0 \\ &\frac{\partial \widehat{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \widehat{\theta\theta}}{\partial \theta} + \frac{\partial \widehat{\theta z}}{\partial z} + \frac{2\widehat{r\theta}}{r} = 0 \\ &\frac{\partial \widehat{rz}}{\partial r} + \frac{1}{r}\frac{\partial \widehat{\theta z}}{\partial \theta} + \frac{\partial \widehat{zz}}{\partial z} + \frac{\widehat{rz}}{r} = 0 \end{aligned}\right\}. \quad (1.2)$$

On integrating the first and third of these with respect to θ from $\theta = -\alpha$ to $\theta = \alpha$, we get

$$\left.\begin{aligned} &\frac{\partial}{\partial r}(r\breve{rr}) + \frac{\partial}{\partial z}(r\breve{rz}) = 0 \\ &\frac{\partial}{\partial r}(r\breve{rz}) + \frac{\partial}{\partial z}(r\breve{zz}) = 0 \end{aligned}\right\}. \quad (1.3)$$

* Love, *Mathematical Theory of Elasticity*, 207. (1927.)

† Ibid., p. 90.

The second of equations (1.2) gives no information about $\breve{rr}$, $\breve{zz}$, or $\breve{rz}$. The equations (1.3) are satisfied if the mean stresses are derived from a stress function χ by the equations

$$\breve{rr} = \frac{1}{r}\frac{\partial^2\chi}{\partial z^2}, \qquad \breve{zz} = \frac{1}{r}\frac{\partial^2\chi}{\partial r^2}, \qquad \breve{rz} = -\frac{1}{r}\frac{\partial^2\chi}{\partial r\partial z}. \tag{1.4}$$

2. The differential equation satisfied by the stress function

To find the differential equation satisfied by the stress function, we use the stress-strain relations.

The mean displacements are defined by the equations

$$U_r = \frac{1}{2\alpha}\int_{-\alpha}^{+\alpha} u_r\, d\theta, \qquad U_z = \frac{1}{2\alpha}\int_{-\alpha}^{+\alpha} u_z\, d\theta. \tag{2.1}$$

Then precisely as in the theory of generalized plane stress* for a plate of constant thickness, we obtain the equations

$$2\mu\frac{\partial U_r}{\partial r} = \breve{rr} - \sigma(\breve{rr}+\breve{zz}), \tag{2.21}$$

$$2\mu\frac{\partial U_z}{\partial z} = \breve{zz} - \sigma(\breve{rr}+\breve{zz}), \tag{2.22}$$

$$\mu\left(\frac{\partial U_r}{\partial z}+\frac{\partial U_z}{\partial r}\right) = \breve{rz}, \tag{2.23}$$

where $1-\sigma = 1/(1+\eta)$ and η is Poisson's ratio.

If we denote $\dfrac{\partial^2\chi}{\partial r^2}+\dfrac{\partial^2\chi}{\partial z^2}$ by $\vartheta^2\chi$, the equations (2.21), (2.22), (2.23) give, with (1.4),

$$2\mu\frac{\partial U_r}{\partial r} = (1-\sigma)\frac{1}{r}\vartheta^2\chi - \frac{1}{r}\frac{\partial^2\chi}{\partial r^2}, \tag{2.31}$$

$$2\mu\frac{\partial U_z}{\partial z} = (1-\sigma)\frac{1}{r}\vartheta^2\chi - \frac{1}{r}\frac{\partial^2\chi}{\partial z^2}, \tag{2.32}$$

$$\mu\left(\frac{\partial U_r}{\partial z}+\frac{\partial U_z}{\partial r}\right) = -\frac{1}{r}\frac{\partial^2\chi}{\partial r\partial z}. \tag{2.33}$$

These equations must be consistent, and so, by eliminating U_r and U_z, we obtain the differential equation satisfied by χ, which is found to be

$$\vartheta^2\left(\frac{1}{r}\vartheta^2\chi\right) = \frac{2(1+\eta)}{r^3}\frac{\partial^2\chi}{\partial z^2}. \tag{2.4}$$

* Coker and Filon, *Photo-Elasticity*, 129. (Camb. 1932.)

3. The displacements

If in (2.32) we put $$\frac{1}{r}\vartheta^2\chi = \frac{\partial^2\phi}{\partial z^2}, \tag{3.1}$$

and integrate with respect to z, we get

$$2\mu U_z = (1-\sigma)\frac{\partial\phi}{\partial z} - \frac{1}{r}\frac{\partial\chi}{\partial z} + f(r), \tag{3.21}$$

where $f(r)$ is an arbitrary function.

To obtain U_r, substitute for U_z from (3.21) in (2.33) and integrate with respect to z. We then obtain

$$2\mu U_r = -(1-\sigma)\frac{\partial\phi}{\partial r} - \frac{1}{r}\frac{\partial\chi}{\partial r} - \frac{\chi}{r^2} - zf'(r) - g'(r), \tag{3.22}$$

where $g'(r)$ is an arbitrary function, written for convenience in this form.

These values of the displacements must also satisfy (2.31), and hence we have the further condition

$$(1-\sigma)\vartheta^2\phi = \frac{2\chi}{r^3} - zf''(r) - g''(r),$$

or $$(1-\sigma)\vartheta^2[\phi + (1+\eta)\{zf(r)+g(r)\}] = \frac{2\chi}{r^3}. \tag{3.23}$$

If we write $$\psi = \phi + (1+\eta)\{zf(r)+g(r)\},$$

we then have the displacements given by

$$2\mu U_r = -(1-\sigma)\frac{\partial\psi}{\partial r} - \frac{1}{r}\frac{\partial\chi}{\partial r} - \frac{\chi}{r^2}, \tag{3.31}$$

$$2\mu U_z = (1-\sigma)\frac{\partial\psi}{\partial z} - \frac{1}{r}\frac{\partial\chi}{\partial z}, \tag{3.32}$$

where ψ is defined by $$\frac{\partial^2\psi}{\partial z^2} = \frac{1}{r}\vartheta^2\chi \tag{3.4}$$

and $$\vartheta^2\psi = \frac{2(1+\eta)\chi}{r^3}. \tag{3.5}$$

It is easily shown that a function ψ can be found which satisfies (3.4) and (3.5). In this way ψ is defined except for terms of the form $ar+bz+crz$, which correspond to states of zero mean stress.

4. The solution of the differential equation for the stress function

4.1. *Solutions independent of the elastic constants.*

If χ is independent of η, we must have

$$\vartheta^2\left\{\frac{1}{r}\vartheta^2\chi\right\} = 0 \tag{4.11}$$

and

$$\frac{\partial^2\chi}{\partial z^2} = 0 \tag{4.12}$$

satisfied simultaneously.

Equation (4.12) gives

$$\chi = R_1 + R_2 z, \tag{4.13}$$

where R_1 and R_2 are functions of r only.

On substituting in (4.11) we find that R_1 and R_2 satisfy the equation

$$\frac{d^2}{dr^2}\left\{\frac{1}{r}\frac{d^2R}{dr^2}\right\} = 0.$$

The solution of this is

$$R = ar^4 + br^3 + cr, \tag{4.14}$$

where a, b, and c are constants. No constant term is included as it gives rise to no mean stresses.

We then obtain the solutions

$$\chi_1 = ar^4 + br^3, \tag{4.15}$$

no term in cr being included, as it gives rise to no mean stresses, and

$$\chi_2 = (ar^4 + br^3 + cr)z. \tag{4.16}$$

These solutions are discussed in § 5.

4.2. *Solutions in the form of power series in r.*

Let us assume that

$$\chi = \sum_{n=0}^{\infty} Z_n \frac{r^n}{n!}, \tag{4.21}$$

where the Z_n are functions of z only.

On substituting this value of χ in (2.4) and equating the coefficients of powers of r to zero, we obtain the results

$$\begin{aligned} Z_0 &= az^2 + bz^3, \\ Z_1 &= c + dz, \\ Z_2 &= \eta(2a + 6bz), \\ \frac{Z_{n+5}}{n+3} + \frac{2Z''_{n+3}[(n+2)^2 - (1+\eta)]}{(n+1)(n+2)(n+3)} + \frac{Z^{(\mathrm{iv})}_{n+1}}{n+1} &= 0, \end{aligned} \tag{4.22}$$

where $n = 0, 1, 2, \ldots$.

It follows that Z_3 and Z_4 are arbitrary and independent of Z_0, Z_1, and Z_2. The term $(c+dz)r$ is included in the solution already obtained in (4.1) and so is omitted.

We then get a solution

$$\chi = az^2+bz^3+\eta(a+3bz)r^2,$$

or, since a and b are independent,

$$\chi_3 = z^2+\eta r^2, \tag{4.23}$$

$$\chi_4 = z^3+3\eta z r^2. \tag{4.24}$$

The remaining terms give a solution

$$\chi = \sum_{n=3}^{\infty} Z_n \frac{r^n}{n!}, \tag{4.25}$$

where Z_3 and Z_4 are arbitrary and the succeeding terms are given by (4.22). The question of the convergence of this series is not considered, as only terminating series will be used in this paper. As examples, let us take some simple values for Z_3 and Z_4.

If $Z_3 = 60z^2, \quad Z_4 = 0,$

then $\chi_5 = 10r^3z^2-(3-\eta)r^5.$

If $Z_3 = 60z^3, \quad Z_4 = 0,$

then $\chi_6 = 10r^3z^3-3(3-\eta)r^5z.$

If $Z_3 = 0, \quad Z_4 = 1080z^2,$

then $\chi_7 = 45r^4z^2-(8-\eta)r^6.$

If $Z_3 = 0, \quad Z_4 = 360z^3,$

then $\chi_8 = 15r^4z^3-(8-\eta)r^6z.$

4.3. *Solutions involving logarithms.*

We have obtained solutions involving the two arbitrary functions Z_3 and Z_4, but since the equation is of the fourth order, we expect to find solutions involving four arbitrary functions.

By analogy with the method of solution of ordinary differential equations in series, we assume

$$\chi = \chi_0 \log r + \sum_{n=0}^{\infty} \zeta_n r^n,$$

where χ_0 is a solution of (2.4) obtained from the functions Z_3 and Z_4, and the ζ_n are functions of z only.

We then substitute in (2.4) and evaluate the ζ_n by equating the coefficients of powers of r to zero.

As an example of this type of solution we find that if

$$\chi_0 = \chi_5,$$

another solution is

$$\chi_9 = \log r\chi_0 + \frac{67-2\eta-39\eta^2}{2(3-\eta)(1+\eta)}z^2r^3 - \frac{5}{2(1+\eta)}z^4r.$$

Further solutions, corresponding to χ_6, χ_7, and χ_8, may easily be obtained by this method.

5. Applications of the solutions

5.1. *Simple tension, Euler-Bernoulli flexure, De Saint Venant flexure.*

In equation (4.15) we have the solution

$$\chi_1 = ar^4+br^3. \tag{5.11}$$

The stresses are

$$\breve{rr} = 0, \qquad \breve{rz} = 0, \qquad \breve{zz} = 12ar+6b. \tag{5.12}$$

The corresponding value of ψ is

$$\psi = 3(2ar+b)z^2-2ar^3-3br^2+\tfrac{1}{3}(1+\eta)(ar^3+3br^2).$$

It follows that the mean displacements are given by

$$\left.\begin{aligned} U_r &= -6E^{-1}\{a(z^2+\eta r^2)+\eta br\} \\ U_z &= 6E^{-1}(2ar+b)z \end{aligned}\right\}, \tag{5.13}$$

where E is Young's modulus.

When applied to a strip with edges $r = \beta$ and $r = \gamma$ this solution corresponds to uniform tension and to bending by terminal couples, and so is of no special interest.

Now, in equation (4.16), we have the solution

$$\chi_2 = (ar^4+br^3+cr)z. \tag{5.14}$$

The stresses are

$$\left.\begin{aligned} \breve{rr} &= 0 \\ \breve{rz} &= -(4ar^2+3br+c/r) \\ \breve{zz} &= (12ar+6b)z \end{aligned}\right\}. \tag{5.15}$$

The function ψ is

$$\psi = (2ar+b)z^3+z\{(\tfrac{1}{3}ar^3+br^2-2c\log r)(1+\eta)-2ar^3-3br^2\},$$

and hence the displacements are

$$\begin{aligned} U_r &= -E^{-1}\{2az^3+6\eta rz(ar+b)\} \\ U_z &= E^{-1}\{[(2ar+b)3z^2-2ar^3-3br^2]-(1+\eta)[\tfrac{2}{3}ar^3+c(1+2\log r)]\}. \end{aligned} \tag{5.16}$$

This solution may be applied to a strip with edges $r = \beta$ and $r = \gamma$. The constants a, b, c can be so chosen as to make these edges free from traction. We then have the solution for De Saint Venant flexure or bending by a transverse load at one end.

The distribution of stress is different from that in a strip of uniform thickness owing to the term c/r in $\breve{rz}$, but $\breve{zz}$ is of the same form.

5.2. *A strip bounded by $r = a$ and $r = b$ and under normal pressure on these edges.*

In equation (4.23) we have the solution

$$\chi_3 = z^2 + \eta r^2. \tag{5.21}$$

The stresses are

$$\breve{rr} = \frac{2}{r}, \qquad \breve{rz} = 0, \qquad \breve{zz} = \frac{2\eta}{r}. \tag{5.22}$$

If $b = 0$, we have the case of a knife pressed by a uniform load on the back $r = a$ against a plane object to be cut. There are, however, tractions across the ends which we may suppose to be $z = \pm c$. It is not possible to annul these tractions completely by any of the stress functions found, but we can superpose a system derived from χ_1 which will make the unwanted tractions at each end form a system in statical equilibrium. The effect of such a system, except near the ends, may be neglected.

The tractions over the end $z = c$, due to χ_3, are equivalent to a force $4\alpha a\eta$ in the line $r = 0$ and a couple of moment* $2\alpha a^2\eta$.

The stress function required to annul this force and couple is

$$\chi = (\eta r^4 - 2\eta a r^3)/a^2.$$

The complete stress function is

$$\chi = z^2 + \eta r^2 + (\eta r^4/a^2) - (2\eta r^3/a), \tag{5.23}$$

and the stresses are

$$\breve{rr} = \frac{2}{r}, \qquad \breve{rz} = 0, \qquad \breve{zz} = \frac{2\eta}{r} + \frac{12\eta r}{a^2} - \frac{12\eta}{a}. \tag{5.24}$$

5.3. *A strip bounded by $r = 0$, $r = a$ under traction on $r = a$.*

The solutions of the type (4.25) are applicable to this problem since all give zero traction on $r = 0$.

First consider a beam under uniform normal load on $r = a$ and supported by shearing stress across the ends $z = \pm c$.

* This is only approximate. An exact value could only be obtained if the complete, instead of mean, distribution of stress were known.

The stresses due to the stress function

$$\chi = \chi_7 - 6a\chi_5 \tag{5.31}$$

are

$$\left.\begin{aligned} \breve{rr} &= -30(4ar^2 - 3r^3) \\ \breve{zz} &= -30[12az^2 - 18rz^2 + 4(3-\eta)ar^2 + (8-\eta)r^3] \\ \breve{rz} &= 360zr(a-r) \end{aligned}\right\}. \tag{5.32}$$

$$\text{On } r = a \qquad \breve{rr} = -30a^3, \qquad \breve{rz} = 0. \tag{5.33}$$

We have then a uniform normal load on $r = a$, and at $z = \pm c$ we have a set of normal tractions and shears. The shears at $z = c$ are statically equivalent to a force of magnitude* $60\alpha ca^4$ in the direction of r increasing. This force is equal and opposite to half the load on $r = a$. The normal tractions may, as in (5.23), be combined with a set of tractions derived from χ_1 to give a system in statical equilibrium.

We can of course combine the solutions of (5.2) and (5.3) to give that for a knife pressed by transverse forces at the ends against the object to be cut.

5.4. *A uniformly loaded cantilever.*

The solution of (5.3) can be applied to the case of a cantilever free at the end $z = 0$ and clamped at the end $z = c$, provided that we again add suitable terms to the stress function to annul the force and couple resultants across $z = 0$.

5.5. *Other soluble problems.*

By suitably choosing Z_3 and Z_4, we can obtain the stress functions corresponding to other distributions of tractions on the edge $r = a$ when the edge $r = 0$ is free from traction.

If we require either a distribution of normal traction on the edge $r = 0$ proportional to z^2 or a higher power, or of tangential traction proportional to z or a higher power, it is necessary to use the stress functions involving $\log r$, of which χ_9 is an example.

In order to find the complete solution for any system of traction on the two edges of such a strip, it would be necessary to find the solutions of (2.4) in the form $F(r)\cos mz$. The functions $F(r)$ so found are, however, so unmanageable as to make such an investigation unprofitable.

* This is again only approximate.

ON THE DISCONTINUOUS INTEGRAL OF WEBER AND SCHAFHEITLIN

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1. Introduction

THE Weber-Schafheitlin integral, namely,

$$I \equiv \int_0^\infty J_\mu(t) J_\nu(ut) t^{-\lambda}\, dt, \tag{1.1}$$

where the parameter u is positive, has been the subject of many papers.* Quite recently, Dixon and Ferrar† have shown how the discontinuity of this integral, according as $u >, =,$ or < 1, may be clearly demonstrated by applying Cauchy's theorem on contour integration to a similar integral involving Bessel functions of imaginary argument.

In the present paper, a new method of evaluating the integral (1.1) is given. From this, it appears that its discontinuity, whether in actual value or in the form of expressing that value, as the case may be, is directly dependent on the discontinuity in value of

$$\int_{c-\infty i}^{c+\infty i} e^{\theta x} x^{-\alpha}\, dx,$$

according as $\theta >, =,$ or < 0. In fact, when there is discontinuity in the actual value of I, namely, when $-1 < R(\lambda) \leqslant 0$ and $\mu - \nu$ is an odd integer, the two discontinuities are of precisely the same nature.

2. The fundamental formulae

When u is positive, the integral I in (1.1) converges provided that

$$R(\mu+\nu+1) > R(\lambda) > -1 \qquad (u \neq 1),$$
$$R(\mu+\nu+1) > R(\lambda) > 0 \qquad (u = 1).$$

We shall find it necessary to restrict still further the parameters λ, μ, ν in order that certain transformations of integrals may be carried out, but shall appeal to the theory of analytical continuation to show that the values of I so obtained are valid whenever I converges.

* See, for example, Watson, *Bessel Functions*, 398–410. (Cambridge 1922.)

† *Quart. J. of Math.* (Oxford), 1 (1930), 122–45.

We replace $J_\nu(ut)$ in I by a modified form of Hankel's contour integral, namely,

$$J_\nu(ut) = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i}\left(\frac{t}{2u}\right)^\nu \exp\left(u^2x - \frac{t^2}{4x}\right)\frac{dx}{x^{\nu+1}},$$

where the path of integration is the straight line $R(x) = c > 0$; this formula is valid when $R(\nu+1) > 0$. Next, we invert the order of integration, to obtain

$$I = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} dx\,\frac{e^{u^2x}}{2^\nu u^\nu x^{\nu+1}}\int_0^\infty t^{\nu-\lambda}J_\mu(t)e^{-t^2/4x}\,dt.$$

Sufficient conditions for the validity of this process are that

$$R(\mu+\nu+1) > R(\lambda) > \tfrac{1}{2}, \qquad R(\nu) > 0,$$

which ensure that the repeated integral converges absolutely.

But, under these same conditions, the integral with respect to t may be evaluated by means of Hankel's generalization* of Weber's first exponential integral, so that

$$I = \frac{1}{2\pi i}\,\frac{\Gamma(\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2}-\frac{1}{2}\lambda)}{2^\lambda u^\nu\Gamma(\mu+1)}\int_{c-\infty i}^{c+\infty i}\frac{e^{u^2x}}{x^{\frac{1}{2}(\nu+1+\lambda-\mu)}}\,{}_1F_1\left(\frac{\mu+\nu+1-\lambda}{2};\mu+1;-x\right)dx \tag{2.1}$$

$$= \frac{1}{2\pi i}\,\frac{\Gamma(\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2}-\frac{1}{2}\lambda)}{2^\lambda u^\nu\Gamma(\mu+1)}\int_{c-\infty i}^{c+\infty i}\frac{e^{(u^2-1)x}}{x^{\frac{1}{2}(\nu+1+\lambda-\mu)}}\,{}_1F_1\left(\frac{\lambda+\mu+1-\nu}{2};\mu+1;x\right)dx, \tag{2.2}$$

the second of these integrals being derived from the first by Kummer's transformation.† The formulae (2.1), (2.2) are fundamental in our subsequent work.

Now, by Barnes's asymptotic expansion‡ of the generalized hypergeometric function, we see that, when $x = c \pm iy$, where y is large and positive,

$$\frac{\Gamma(\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2}-\frac{1}{2}\lambda)e^{u^2x}}{2^\lambda u^\nu\Gamma(\mu+1)x^{\frac{1}{2}(\nu+\lambda+1-\mu)}}\,{}_1F_1\left(\frac{\mu+\nu+1-\lambda}{2};\mu+1;-x\right)$$

$$\sim \frac{(-1)^{\frac{1}{2}(\nu-\lambda-\mu-1)}e^{(u^2-1)x}}{2^\lambda u^\nu x^{1+\lambda}} + \frac{\Gamma(\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2}-\frac{1}{2}\lambda)e^{u^2x}}{\Gamma(\frac{1}{2}\lambda+\frac{1}{2}\mu+\frac{1}{2}-\frac{1}{2}\nu)2^\lambda u^\nu x^{1+\nu}}; \tag{2.3}$$

hence the integral (2.1) converges when $R(\mu+\nu+1) > R(\lambda) > -1$, $R(\nu) > -1$, $(u \neq 1)$ or when $R(\mu+\nu+1) > R(\lambda) > 0$, $R(\nu) > -1$,

* See Watson, loc. cit., 393, (2).

† Viz. ${}_1F_1(\alpha;\rho;x) = e^x\,{}_1F_1(\rho-\alpha;\rho;-x)$. See Watson, loc. cit., 102.

‡ *Cambridge Phil. Trans.* 20 (1908), 259.

($u = 1$). It can be shown, by analytical continuation, that the equations (2.1) and (2.2) still hold under these less restrictive conditions.

The asymptotic formula (2.3) also shows that, when $u = 1$, $R(\lambda) = 0$, the integral in (2.1) oscillates finitely, save when $\lambda = 0$, though in the latter case it exists as a Cauchy principal value.

3. The general case

In certain particular cases, the generalized hypergeometric function which occurs in (2.2) is a polynomial, a fact which then simplifies the evaluation of the integral. In the general case, we use the formula

$$\frac{\Gamma(\alpha)\Gamma(\rho-\alpha)}{\Gamma(\rho)}\,{}_1F_1(\alpha;\rho;x) = \int_0^1 e^{x(1-t)}t^{\rho-\alpha-1}(1-t)^{\alpha-1}\,dt$$

which holds when this integral converges.

If we suppose, temporarily,* that

$$R(\mu+\nu+1) > R(\lambda) > |R(\mu-\nu+1)|,$$

we obtain in this way from (2.2) the equation

$$\begin{aligned} I &= \frac{1}{2^{\lambda+1}\pi i u^{\nu}\Gamma(\tfrac{1}{2}\lambda+\tfrac{1}{2}\mu+\tfrac{1}{2}-\tfrac{1}{2}\nu)}\times \\ &\qquad \times \int_{c-\infty i}^{c+\infty i} dx \int_0^1 \frac{e^{x(u^2-t)}}{x^{\frac{1}{2}(\nu+\lambda+1-\mu)}}\,t^{\frac{1}{2}(\mu+\nu-\lambda-1)}(1-t)^{\frac{1}{2}(\lambda+\mu-\nu-1)}\,dt \\ &= \frac{1}{2^{\lambda+1}\pi i u^{\nu}\Gamma(\tfrac{1}{2}\lambda+\tfrac{1}{2}\mu+\tfrac{1}{2}-\tfrac{1}{2}\nu)}\times \\ &\qquad \times \int_0^1 dt\, t^{\frac{1}{2}(\mu+\nu-\lambda-1)}(1-t)^{\frac{1}{2}(\lambda+\mu-\nu-1)} \int_{c-\infty i}^{c+\infty i} \frac{e^{x(u^2-t)}}{x^{\frac{1}{2}(\nu+\lambda+1-\mu)}}\,dx, \quad (3.1) \end{aligned}$$

the inversion of order being justified by the absolute convergence of the repeated integral.

Now, since $R(\nu+\lambda-\mu-1) > 0$, we have†

$$\frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} \frac{e^{x(u^2-t)}}{x^{\frac{1}{2}(\nu+\lambda+1-\mu)}}\,dx = \frac{(u^2-t)^{\frac{1}{2}(\nu+\lambda-\mu-1)}}{\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}\lambda+\tfrac{1}{2}-\tfrac{1}{2}\mu)} \text{ or } 0,$$

according as $u^2 >$ or $\leqslant t$; it is from this fact, as we shall now show, that the discontinuity in form of the value of I arises.

* We could have avoided these restrictions by the use of the Pochhammer double-circuit integral for ${}_1F_1$.

† Cf. Whittaker and Watson, *Modern Analysis*, 246, Ex. 1. (Cambridge 1920.)

When $u > 1$, the inequality $u^2 > t$ is satisfied over the whole range of integration with respect to t, so that, in this case,

$$I = \frac{1}{2^\lambda u^\nu \Gamma(\frac{1}{2}\lambda+\frac{1}{2}\mu+\frac{1}{2}-\frac{1}{2}\nu)\Gamma(\frac{1}{2}\nu+\frac{1}{2}\lambda+\frac{1}{2}-\frac{1}{2}\mu)} \times$$

$$\times \int_0^1 (u^2-t)^{\frac{1}{2}(\nu+\lambda-\mu-1)} t^{\frac{1}{2}(\mu+\nu-\lambda-1)} (1-t)^{\frac{1}{2}(\lambda+\mu-\nu-1)}\, dt \quad (3.2)$$

$$= \frac{u^{\lambda-\mu-1}\Gamma(\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2}-\frac{1}{2}\lambda)}{2^\lambda \Gamma(\frac{1}{2}\nu+\frac{1}{2}\lambda+\frac{1}{2}-\frac{1}{2}\mu)\Gamma(\mu+1)} \times$$

$$\times F\left(\frac{\mu+\nu+1-\lambda}{2}, \frac{\mu+1-\nu-\lambda}{2}; \mu+1; \frac{1}{u^2}\right), \quad (3.3)$$

by a well-known integral* representing the hypergeometric function. The restrictions under which (3.3) has been proved may now be relaxed by the use of the theory of analytical continuation; in fact (3.3) holds for $u > 1$, whenever I converges.

On the other hand, when $u \leqslant 1$, the inequality $u^2 > t$ is not satisfied over the whole range of integration with respect to t, so that we now have

$$I = \frac{1}{2^\lambda u^\nu \Gamma(\frac{1}{2}\lambda+\frac{1}{2}\mu+\frac{1}{2}-\frac{1}{2}\nu)\Gamma(\frac{1}{2}\nu+\frac{1}{2}\lambda+\frac{1}{2}-\frac{1}{2}\mu)} \times$$

$$\times \int_0^{u^2} (u^2-t)^{\frac{1}{2}(\nu+\lambda-\mu-1)} t^{\frac{1}{2}(\mu+\nu-\lambda-1)} (1-t)^{\frac{1}{2}(\lambda+\mu-\nu-1)}\, dt. \quad (3.4)$$

When $u < 1$, we make the transformation $t = \tau u^2$ in (3.4) and obtain

$$I = \frac{u^\nu \Gamma(\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2}-\frac{1}{2}\lambda)}{2^\lambda \Gamma(\frac{1}{2}\lambda+\frac{1}{2}\mu+\frac{1}{2}-\frac{1}{2}\nu)\Gamma(\nu+1)} \times$$

$$\times F\left(\frac{\mu+\nu+1-\lambda}{2}, \frac{\nu+1-\lambda-\mu}{2}; \nu+1; u^2\right), \quad (3.5)$$

a result which still subsists whenever I converges.

Finally, when $u = 1$, the integral in (3.4) is a case of the first Eulerian integral, so that

$$I = \frac{\Gamma(\lambda)\Gamma(\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2}-\frac{1}{2}\lambda)}{2^\lambda \Gamma(\frac{1}{2}\lambda+\frac{1}{2}\mu+\frac{1}{2}-\frac{1}{2}\nu)\Gamma(\frac{1}{2}\nu+\frac{1}{2}\lambda+\frac{1}{2}-\frac{1}{2}\mu)\Gamma(\frac{1}{2}\lambda+\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2})}, \quad (3.6)$$

In this case, too, we may show, by analytical continuation, that (3.6) still represents I, when the only restrictions on the parameters are that I should converge.

* Whittaker and Watson, loc. cit., 293, Ex. 1.

Now when $R(\mu+\nu+1) > R(\lambda) > 0$, I converges for all positive values of u, and a comparison of (3.3), (3.5), (3.6) shows that it is continuous at $u = 1$. The form of the value of I is, however, discontinuous, since the value for $u > 1$ is not the analytical continuation of the value when $u < 1$. There is, however, a genuine discontinuity when $-1 < R(\lambda) \leqslant 0$, since the integral does not then converge for all values of u.

4. The critical case

It follows from the asymptotic expansion of the Bessel function that the integral (1.1) converges for all positive values of u, when $\nu-\mu$ is an odd integer, provided that λ lies in the strip

$$-1 < R(\lambda) < R(\mu+\nu+1);$$

further, for any fixed value of u, the integral is an analytic function of λ, regular in this strip.

Now in this particular case, the expression (3.6) is also a regular function of λ in the same strip, the singularity at $\lambda = 0$ being a removable one. Thus, by analytical continuation, the formula (3.6), or its limiting form as $\lambda \to 0$, gives the value of I, when $\nu-\mu$ is an odd integer, in the extended range $-1 < R(\lambda) \leqslant 0$.

But when we regard the Weber-Schafheitlin integral as a function of u, the discontinuity of form becomes an actual discontinuity in this critical case. Although this discontinuity may be demonstrated* on the lines indicated above, it seems to be of some interest to show how it arises by the direct method of the present paper.

Let us consider, then, the integral

$$J \equiv \int_0^\infty J_{\alpha-p-1}(t) J_{\alpha+p}(ut)\, dt, \tag{4.1}$$

where p is a positive integer or zero; it converges when $R(\alpha) > 0$. It follows from (2.2) that

$$\begin{aligned} J &= \frac{\Gamma(\alpha)}{2\pi i\, u^{\alpha+p}\Gamma(\alpha-p)} \int_{c-\infty i}^{c+\infty i} \frac{e^{(u^2-1)x}}{x^{1+p}}\, {}_1F_1(-p;\alpha-p;x)\, dx \\ &= \frac{\Gamma(\alpha).p!}{2\pi i\, u^{\alpha+p}\Gamma(\alpha-p)\Gamma(1-\alpha+p)} \int_{c-\infty i}^{c+\infty i} e^{(u^2-1)x} \sum_{r=0}^{p} \frac{\Gamma(1-\alpha+r)}{r!(p-r)!\, x^{1+r}}\, dx, \end{aligned} \tag{4.2}$$

the integral being a Cauchy principal value when $u = 1$.

* See Dixon and Ferrar, loc. cit., 132.

Now, when $u^2 < 1$ and s is a positive integer,

$$\int_{c-\infty i}^{c+\infty i} \frac{e^{(u^2-1)x}}{x^s}\,dx = 0,$$

so that J vanishes when $u < 1$. But when $u = 1$,

$$J = \frac{1}{2\pi i}\frac{\Gamma(\alpha)\Gamma(1-\alpha)}{\Gamma(\alpha-p)\Gamma(1-\alpha+p)}\lim_{R\to\infty}\int_{c-Ri}^{c+Ri}\frac{dx}{x} = \tfrac{1}{2}(-1)^p.$$

Finally, when $u^2 > 1$, we have,* by (2.1),

$$\begin{aligned} J &= \frac{\Gamma(\alpha)}{2\pi i\, u^{\alpha+p}\Gamma(\alpha-p)}\int_{-\infty}^{(0+)}\frac{e^{u^2x}}{x^{1+p}}\,{}_1F_1(\alpha;\alpha-p;-x)\,dx \\ &= \frac{1}{2\pi i\, u^{\alpha+p}}\int_{-\infty}^{(0+)}\frac{e^{u^2x}}{x^{1+p}}\sum_{r=0}^{\infty}\frac{\Gamma(\alpha+r)}{\Gamma(\alpha-p+r)}\frac{(-x)^r}{r!}\,dx \\ &= \frac{\Gamma(\alpha)}{u^{\alpha-p}\Gamma(\alpha-p).p!}F\left(\alpha,-p;\alpha-p;\frac{1}{u^2}\right). \end{aligned}$$

We may, then, sum up the results of this section in the formula

$$\int_0^\infty J_{\alpha+p}(ut)J_{\alpha-p-1}(t)\,dt = \begin{cases} \dfrac{\Gamma(\alpha)}{u^{\alpha-p}\Gamma(\alpha-p).p!}F\left(\alpha,-p;\alpha-p;\dfrac{1}{u^2}\right), \\ \tfrac{1}{2}(-1)^p, \\ 0, \end{cases}$$

according as $u >$, $=$, or < 1, provided that $R(\alpha) > 0$.

* It should be observed that this change of contour may be applied to (2.1) in the general case and that it leads directly to (3.3).

DU BOIS REYMOND'S CONSTANTS

By G. N. WATSON (*Birmingham*)

[Received 6 April 1933]

THE constants which I shall discuss in this paper have their origin in the following theorem:

Let $\sum_{n=1}^{\infty} a_n$ *be an oscillating series whose upper and lower limits, U and L, are finite. Let* $\phi(t)$ *be a function with a differential coefficient which is finite when* $t \geqslant 0$, *such that*

$$\lim_{t \to +0} \phi(t) = 1, \qquad \int_0^{\infty} \left| \frac{d\phi(t)}{dt} \right| dt < \infty.$$

Then, if the series $\sum_{n=1}^{\infty} a_n \phi(nt)$ *converges whenever* $t > 0$ *and has the sum* $S(t)$, *the upper and lower limits of* $S(t)$ *as* $t \to +0$ *lie between the two numbers*

$$\tfrac{1}{2}(U+L) \pm \tfrac{1}{2}(U-L) \int_0^{\infty} \left| \frac{d\phi(t)}{dt} \right| dt.$$

The history of the theorem is as follows: in Du Bois Reymond's work* on trigonometrical series, the series $\sum a_n$ was a trigonometrical series with $a_n = o(1)$ and $\phi(t)$ had the special value $t^{-2}\sin^2 t$. Du Bois Reymond obtained the upper and lower limits

$$\tfrac{1}{2}(U+L) \pm \tfrac{1}{2}(U-L)(1+c),$$

where

$$c \leqslant 2 + \frac{2}{\pi} + \frac{2}{\pi^2},$$

by an *ad hoc* argument. The type of argument which he used was apparently sufficient† to prove that

$$c \leqslant \frac{1}{\pi} + \frac{1}{\pi^2},$$

and the latter constant had, in fact, appeared in 1854 in Riemann's *Habilitationsschrift*,‡ where it was assumed that $\sum a_n$ was convergent so that $U = L$.

* *Abh. der Kgl. Bayerischen Akad. zu München*, 12 (1876), 119–66.

† Hobson, *Functions of a Real Variable* (1907), 732–4.

‡ See his *Ges. Math. Werke* (1892), 227–64 (247).

Later, by an argument which is applicable to a general function $\phi(t)$, G. C. Young* proved that, when $\phi(t) = t^{-2}\sin^2 t$,

$$1+c = \int_0^\infty \left| \frac{d}{dt}\left(\frac{\sin^2 t}{t^2}\right)\right| dt < 1+\frac{2}{\pi^2},$$

and the theorem in its general form was actually given by Hobson.†

Since Du Bois Reymond's work exhibits the importance of the numerical value of c, it seems appropriate to associate his name, rather than Riemann's,‡ with such a constant, and to describe

$$\int_0^\infty \left| \frac{d\phi(t)}{dt}\right| dt - 1$$

as Du Bois Reymond's constant associated with $\phi(t)$.

In recent work on trigonometrical series,§ derived series in which $\phi(t)$ has been given the values $\left(\frac{\sin t}{t}\right)^3$ and $\left(\frac{\sin t}{t}\right)^4$ have been the subject of investigation. It consequently seems worth while to examine the constants c_m defined by the formula

$$c_m = \int_0^\infty \left| \frac{d}{dt}\left(\frac{\sin t}{t}\right)^m\right| dt - 1.$$

I shall show that c_{2m} is expressible as a polynomial of degree m in e^2 with rational coefficients.‖ There seems to be no similar simple formula for c_{2m+1}, and the sum total of the results of my analytical attacks on c_3 has yielded merely an integral which does not lend itself readily to numerical computation plus a series whose sum is fairly easy to compute; I have, however, found it easy to compute c_3 by elementary methods.

If $$\phi(t) = \left(\frac{\sin t}{t}\right)^{2m},$$

it is easy to see that the minima of $\phi(t)$ in the interval $(0, \infty)$ occur

* *Messenger of Math.* 49 (1920), 73–8.

† *Functions of a Real Variable*, 2 (1926), 221–4.

‡ Obviously all that matters in Riemann's work is that c should be *finite*; its numerical value is irrelevant.

§ Verblunsky, *Proc. London Math. Soc.* (2) 31 (1930), 387–406; *Journal London Math. Soc.* 6 (1931), 82–4.

‖ The corresponding coefficients in the expression for $2^{2m-1}(m-1)!c_{2m}$ are integers.

at $t = \pi,\ 2\pi,\ 3\pi,\ldots$ and the maxima occur at $t = 0,\ \varpi_1,\ \varpi_2,\ \varpi_3,\ldots$, where $\varpi_1,\ \varpi_2,\ \varpi_3,\ldots$ are the positive roots of the equation

$$\sin t - t\cos t = 0.$$

Also $$\phi(n\pi) = 0 \qquad (n = 1, 2, 3,\ldots),$$

$$\phi(\varpi_n) = \cos^{2m}\varpi_n = \frac{1}{(1+\varpi_n^2)^m} \qquad (n = 1, 2, 3,\ldots).$$

Hence

$$\begin{aligned} 1+c_{2m} &= \{\phi(0)-\phi(\pi)\}+|\phi(\pi)-\phi(\varpi_1)|+ \\ &\qquad +\{\phi(\varpi_1)-\phi(2\pi)\}+|\phi(2\pi)-\phi(\varpi_2)|+\ldots \\ &= 1+2\sum_{n=1}^{\infty}\phi(\varpi_n), \end{aligned}$$

so that $$c_{2m} = 2\sum_{n=1}^{\infty}\frac{1}{(1+\varpi_n^2)^m}.$$

Now consider the integral

$$\frac{1}{2\pi i}\int \frac{z\sin z}{\sin z - z\cos z}\,\frac{dz}{(1+z^2)^m},$$

where the path of integration is a (large) circle with centre at the origin. As the radius of the circle tends (discontinuously) to infinity in such a way as to avoid the poles of the integrand, the integral tends to zero.

The poles of the integrand are at the origin, where the residue is 3, at the points $\pm\varpi_n$ $(n = 1, 2, 3,\ldots)$, where the residues are $(1+\varpi_n^2)^{-m}$, and at the points $\pm i$.

By putting $z = \pm i\zeta$, we see that the residues at $\pm i$ are both equal to the residue of

$$-\frac{\zeta\sinh\zeta}{\sinh\zeta - \zeta\cosh\zeta}\,\frac{1}{(1-\zeta^2)^m}$$

at $\zeta = 1$, that is, they are equal to

$$\frac{(-)^m}{(m-1)!}\left[\frac{d^{m-1}}{d\zeta^{m-1}}\left\{\frac{\zeta\sinh\zeta}{\zeta\cosh\zeta - \sinh\zeta}\,\frac{1}{(\zeta+1)^m}\right\}\right]_{\zeta=1}.$$

Hence, by Cauchy's theorem,

$$c_{2m} = -3 - \frac{(-)^m 2}{(m-1)!}\left[\frac{d^{m-1}}{d\zeta^{m-1}}\left\{\frac{\zeta\sinh\zeta}{\zeta\cosh\zeta - \sinh\zeta}\,\frac{1}{(\zeta+1)^m}\right\}\right]_{\zeta=1}.$$

Now

$$\frac{\zeta\sinh\zeta}{\zeta\cosh\zeta-\sinh\zeta}=\frac{\zeta(e^{\zeta}-e^{-\zeta})}{(\zeta+1)e^{-\zeta}+(\zeta-1)e^{\zeta}}$$

$$=\zeta(e^{\zeta}-e^{-\zeta})\sum_{r=0}^{\infty}(-)^r\frac{(\zeta-1)^r}{(\zeta+1)^{r+1}}e^{(2r+1)\zeta}$$

$$=-\frac{\zeta}{\zeta+1}+2\sum_{r=1}^{\infty}(-)^{r-1}\frac{\zeta^2(\zeta-1)^{r-1}}{(\zeta+1)^{r+1}}e^{2r\zeta},$$

and therefore

$$\frac{\zeta\sinh\zeta}{\zeta\cosh\zeta-\sinh\zeta}\frac{1}{(\zeta+1)^m}=\frac{1}{(\zeta+1)^{m+1}}-\frac{1}{(\zeta+1)^m}+$$

$$+2\sum_{r=1}^{\infty}(-)^{r-1}(\zeta-1)^{r-1}e^{2r\zeta}\left\{\frac{1}{(\zeta+1)^{m+r+1}}-\frac{2}{(\zeta+1)^{m+r}}+\frac{1}{(\zeta+1)^{m+r-1}}\right\}.$$

Hence

$$(-)^{m-1}\left[\frac{d^{m-1}}{d\zeta^{m-1}}\left\{\frac{\zeta\sinh\zeta}{\zeta\cosh\zeta-\sinh\zeta}\frac{1}{(\zeta+1)^m}\right\}\right]_{\zeta=1}$$

$$=\frac{(2m-1)!}{2^{2m}m!}-\frac{(2m-2)!}{2^{2m-1}(m-1)!}+2\sum_{r=1}^{m}(-)^{m-r}{}_{m-1}C_{r-1}(r-1)!\times$$

$$\times\left[\frac{d^{m-r}}{d\zeta^{m-r}}e^{2r\zeta}\left\{\frac{1}{(\zeta+1)^{m+r+1}}-\frac{2}{(\zeta+1)^{m+r}}+\frac{1}{(\zeta+1)^{m+r-1}}\right\}\right]_{\zeta=1}$$

$$=-\frac{(2m-2)!}{2^{2m}m!}+2\sum_{r=1}^{m}\frac{(m-1)!}{(m-r)!}\left[\sum_{s=0}^{m-r}(-)^s{}_{m-r}C_s(2r)^se^{2r}\times\right.$$

$$\left.\times\left\{\frac{(2m-s)!}{2^{2m-s+1}(m+r)!}-2\frac{(2m-s-1)!}{2^{2m-s}(m+r-1)!}+\frac{(2m-s-2)!}{2^{2m-s-1}(m+r-2)!}\right\}\right].$$

That is to say,

$$c_{2m}=-3-\frac{(2m-2)!}{2^{2m-1}m!(m-1)!}+$$

$$+\sum_{r=1}^{m}\sum_{s=0}^{m-r}\frac{(-)^s(4r)^se^{2r}}{2^{2m-1}s!(m-r-s)!}\left\{\frac{(2m-s)!}{(m+r)!}-4\frac{(2m-s-1)!}{(m+r-1)!}+4\frac{(2m-s-2)!}{(m+r-2)!}\right\},$$

which is the required expression for c_{2m}.

This formula gives the following results:

$$2c_2=e^2-7, \qquad c_2=0{\cdot}19452\,80495,$$
$$8c_4=e^4-4e^2-25, \qquad c_4=0{\cdot}00524\,07047,$$
$$32c_6=e^6-6e^4+3e^2-98, \qquad c_6=0{\cdot}00022\,06747.$$

The value of c_2 should be compared with the value of G. C. Young's approximation $2\pi^{-2} = 0{\cdot}20264$.

Up to a point the investigation of c_3 is very similar to that of c_{2m}. We observe that the maxima of $t^{-3}\sin^3 t$ in $(0,\infty)$ are at $t=0$, ϖ_2, ϖ_4,..., and the minima are at ϖ_1, ϖ_3,..., whence

$$c_3 = 2\sum_{n=1}^{\infty}(-)^n\frac{\sin^3\varpi_n}{\varpi_n^3} = 2\sum_{n=1}^{\infty}\frac{1}{(1+\varpi_n^2)^{\frac{3}{2}}}.$$

Now consider the integral

$$\frac{1}{2\pi i}\int \frac{\sin^2 z}{(\sin z - z\cos z)\cos z}\frac{dz}{\sqrt{(a^2+z^2)}} \qquad (a>0),$$

where the path of integration consists of two (large) equal semicircles on the left and right of the imaginary axis respectively, both having their centres at the origin, the ends of the semicircles being joined by loops to exclude the points $\pm ai$ from the interior of the contour.

The poles of the integrand are at the origin, where the residue is $3/a$, at the points $\pm\varpi_n$, where the residues are $(a^2+\varpi_n^2)^{-\frac{1}{2}}$, and the points $\pm(n-\frac{1}{2})\pi$, where the residues are $-\{a^2+(n-\frac{1}{2})^2\pi^2\}^{-\frac{1}{2}}$.

It follows from Cauchy's theorem that

$$\begin{aligned}
&\frac{3}{a}+2\sum_{n=1}^{\infty}\left\{\frac{1}{\sqrt{(a^2+\varpi_n^2)}}-\frac{1}{\sqrt{\{a^2+(n-\frac{1}{2})^2\pi^2\}}}\right\}\\
&\quad=\frac{1}{2\pi i}\left\{\int_{\infty i}^{(ai-)}+\int_{-\infty i}^{(-ai-)}\right\}\frac{\sin^2 z}{(\sin z - z\cos z)\cos z}\frac{dz}{\sqrt{(a^2+z^2)}}\\
&\quad=-\frac{1}{\pi i}\int_{+\infty}^{(a-)}\frac{\sinh^2\zeta}{(\zeta\cosh\zeta-\sinh\zeta)\cosh\zeta}\frac{d\zeta}{\sqrt{(a^2-\zeta^2)}}\\
&\quad=\frac{2}{\pi}\int_a^{\infty}\frac{\sinh^2 x}{(x\cosh x-\sinh x)\cosh x}\frac{dx}{\sqrt{(x^2-a^2)}}\\
&\quad=\frac{2}{\pi}\int_1^{\infty}\frac{\sinh^2 ax}{(ax\cosh ax-\sinh ax)\cosh ax}\frac{dx}{\sqrt{(x^2-1)}}.
\end{aligned}$$

Now differentiate with respect to a, and put $a=1$ after differentia-

tion; we get

$$3+2\sum_{n=1}^{\infty}\frac{1}{(1+\varpi_n^2)^{\frac{3}{2}}}-2\sum_{n=1}^{\infty}\frac{1}{\{1+(n-\frac{1}{2})^2\pi^2\}^{\frac{3}{2}}}$$

$$=-\frac{2}{\pi}\int_1^{\infty}\left[\frac{d}{da}\frac{\sinh^2 ax}{(ax\cosh ax-\sinh ax)\cosh ax}\right]_{a=1}\frac{dx}{\sqrt{(x^2-1)}}$$

$$=-\frac{2}{\pi}\int_1^{\infty}\frac{d}{dx}\left(\frac{\tanh^2 x}{x-\tanh x}\right)\frac{x\,dx}{\sqrt{(x^2-1)}}.$$

I have not discovered any simple method of computing the integral on the right.

It is possible to compute c_3 directly in the following manner. We have* $\varpi_1 = 4{\cdot}49340\,94579, \qquad \varpi_2 = 7{\cdot}72525\,18369,$

while, if $(n+\frac{1}{2})\pi = \alpha_n$, we have, by a formula due to Euler,†

$$\varpi_n = \alpha_n - \frac{1}{\alpha_n} - \frac{2}{3\alpha_n^3} - \frac{13}{15\alpha_n^5} - \frac{146}{105\alpha_n^7} + \ldots,$$

whence $$\frac{1}{(1+\varpi_n^2)^{\frac{3}{2}}} = \frac{1}{\alpha_n^3} + \frac{3}{2\alpha_n^5} + \frac{19}{8\alpha_n^7} + \frac{323}{80\alpha_n^9} + \frac{19519}{2688\alpha_n^{11}} + \ldots.$$

From this result it is easy to compute

$$\sum_{n=3}^{\infty}\frac{1}{(1+\varpi_n^2)^{\frac{3}{2}}}$$

with the help of Glaisher's tables‡ of

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^m}.$$

This procedure gives $$c_3 = 0{\cdot}02825\,17642.$$

Finally I remark that

$$-2\sum_{n=1}^{\infty}\frac{\sin^3\varpi_n}{\varpi_n^3} = 2\sum_{n=1}^{\infty}\frac{(-)^{n-1}}{(1+\varpi_n^2)^{\frac{3}{2}}}$$

* The values of ϖ_1 and ϖ_2 to four places of decimals are quoted from Lommel by Jahnke u. Emde, *Funktionentafeln* (1923), 3. I computed the values stated directly by interpolation in Andoyer's fifteen-figure trigonometrical tables.

† *Introductio in analysin infinitorum*, 2 (1748), 319; this is a special case of Stokes's formula for zeros of Bessel functions applied to $J_{\frac{1}{2}}(x)$.

‡ *Quart. J. of Math.* 45 (1914), 141–58.

is easily evaluated by considering

$$-\frac{1}{2\pi i}\int \frac{z^2}{\sin z - z\cos z}\frac{dz}{(1+z^2)^2},$$

and it is found to be equal to

$$3-\tfrac{1}{4}(e^3-3e) = 0{\cdot}01732\,71405.$$

While computing c_3, I also computed this constant by the same method and obtained a value differing from the correct value by a unit in the eleventh place of decimals.

It is, of course, possible to compute c_5, c_7,... in the way in which I computed c_3, and it is also possible to obtain integral representations of c_5, c_7,... analogous to the integral for c_3 given above.

Also, by considering the contour integral

$$-\frac{1}{2\pi i}\int \frac{z^2}{\sin z - z\cos z}\frac{dz}{(1+z^2)^{m+1}},$$

it is possible to prove that

$$-2\sum_{n=1}^{\infty}\frac{\sin^{2m+1}\varpi_n}{\varpi_n^{2m+1}} = 2\sum_{n=1}^{\infty}\frac{(-)^{n-1}}{(1+\varpi_n^2)^{m+1}}$$

is expressible as a polynomial of degree $2m+1$ in e with rational coefficients.

ON CRITERIA FOR UNIVERSALITY OF TERNARY QUADRATIC FORMS

By ARNOLD E. ROSS (*Pasadena*)

[Received 8 December 1932]

1. In this paper we derive necessary and sufficient conditions that an indefinite ternary quadratic form

$$\Psi = \alpha x^2+\beta y^2+\gamma z^2+\rho yz+\sigma xz+\tau xy \qquad (1.1)$$

with integral coefficients $\alpha,\ldots,\tau$ should be universal, i.e. should represent all positive and all negative integers. We shall also show very simply that every universal form Ψ represents zero properly. If ρ, σ, τ are all even, form Ψ is called *classic*; if at least one of ρ, σ, τ is odd, it is called *non-classic*. If (1.1) is a properly primitive classic form, conditions for its universality are expressed in terms of its generic characters. If (1.1) is a non-classic form these conditions are given in terms of the generic characters of an improperly primitive classic form $f = 2\Psi$.

Universal ternary quadratic forms have been studied by Dickson* and Oppenheim.† The present writer noticed‡ that if the determinant of a universal classic ternary form is odd or a double of an odd integer, and its first invariant Ω is equal to ± 1, then Meyer's§ criterion for equivalence yields at once necessary and sufficient conditions for universality of a classic form Ψ in terms of its generic characters.

In the present paper we study both classic and non-classic forms and obtain the desired necessary and sufficient conditions without the use of the above-mentioned deep and difficult theorem of Meyer.‖

THEOREM 1. *Let f be a properly primitive, indefinite, classic, ternary, quadratic form with reciprocal F and determinant D. The necessary and sufficient conditions that f be universal are*

$$D \text{ is odd or double an odd integer}, \ \Omega = \mp 1, \qquad (1.2)$$

and, for every odd prime p dividing $\Delta = D$,

$$(F|p) = (-\Omega|p). \qquad (1.3)$$

* Dickson (**2**), (**4**), 21.

† Oppenheim (**6**).

‡ Ross (**7**).

§ Meyer (**5**).

‖ A brief report on the results of this paper appeared in the *Proceedings of the National Academy of Sciences*, September 1932. There is a misprint in the statement of Theorem 2.

THEOREM 2. *Let f_1 be a primitive, indefinite, non-classic, ternary, quadratic form. Consider an improperly primitive form $f = 2f_1$. Let F be the reciprocal and Ω, Δ the invariants of f. Then f_1 is universal if and only if $f = 2f_1$ satisfies the following conditions:*

$$\Omega = \pm 1, \tag{1.4}$$

the characters of f are

$$(-1)^{\frac{1}{2}(F-1)} = (-1)^{-\frac{1}{2}(\Omega-1)}, \qquad (F|p) = (-\Omega|p) \tag{1.5}$$

for every odd prime p dividing $\Delta = D$, and, in case 4 divides Δ,

$$(-1)^{\frac{1}{8}(F^2-1)} = 1. \tag{1.6}$$

2. First lemma. In this section we reduce* the problem of representation of an integer m by a classic form (1.1), of determinant D with α equal to an odd prime or double such a prime, to the problem of representation of αm by a form of type

$$\Phi = x^2 + B(y,z), \tag{2.1}$$

where $B(y,z)$ is a binary quadratic form of determinant αD. We show that if the form (2.1) represents a multiple αm of m, then the original form Ψ represents m.

LEMMA 1. *Consider a classic ternary quadratic form*

$$f = ax^2 + by^2 + cz^2 + 2ryz + 2sxz + 2txy \tag{2.2}$$

with the invariants Ω and Δ. Write

$$ab - t^2 = \Omega C, \qquad ac - s^2 = \Omega B, \qquad ar - st = -\Omega R. \tag{2.3}$$

Multiplying both members of (2.2) by a, we get

$$af = (ax + ty + sz)^2 + (ab - t^2)y^2 + 2(ra - st)yz + (ac - s^2)z^2, \tag{2.4}$$

or, in notations (2.3),

$$af = (ax + ty + sz)^2 + \Omega C y^2 - 2\Omega R yz + \Omega B z^2. \tag{2.5}$$

Next write $\qquad \Phi(X,y,z) = X^2 + \Omega C y^2 - 2\Omega R yz + \Omega B z^2. \tag{2.6}$

Let a be a prime or a double of a prime not dividing determinant $D = \Omega^2\Delta$ of f. Then (i) *if f represents m, Φ represents am, and* (ii), *conversely, if Φ represents am, then f represents m.*

Part (i) of the lemma is obvious. To prove part (ii) we assume that

$$\Phi(X,y,z) = am. \tag{2.7}$$

* Lemma 1 is a particular case of a more general theorem for quadratic forms in n variables and also of another theorem which reduces the problem to that of representation by a sum of squares with integral coefficients. See the above-mentioned report in the *Proceedings*.

By (2.6) and (2.3)

$$\Phi(X,y,z) \equiv X^2-t^2y^2-2styz-s^2z^2 \pmod{a}$$
$$\equiv X^2-(ty+sz)^2 \pmod{a}.$$

Since by (2.7) $\Phi \equiv 0 \pmod{a}$, and since a is a prime or a double of a prime, either $X-ty-sz \equiv 0$ or $-X-ty-sz \equiv 0 \pmod{a}$. Hence there exists an integer x such that $eX = ax+ty+sz$, where $e = +1$ or -1. Substitution of this value of X into (2.7) and relation (2.5) yield the equality

$$am = (ax+ty+sz)^2+\Omega Cy^2-2\Omega Ryz+\Omega Bz^2 = af(x,y,z).$$

Therefore $m = f(x,y,z)$, and, since x, y, z are all integers, f represents m.

3. Second lemma. The proof of our second lemma employs an argument from the concluding part of Oppenheim's* proof of his first theorem.

LEMMA 2. *Consider a form*

$$h = x^2+Ly^2+2Myz+Nz^2 = x^2+\phi(y,z) \tag{3.11}$$

of determinant

$$d = LN-M^2, \text{ where } (L,2M,N) = 1. \tag{3.12}$$

If $(L,d) = 1$ *and* $-L$ *is a quadratic residue of* d, (3.13)

then h *is equivalent to* $2xy+x^2-dz^2$. (3.14)

Multiplying both members of (3.11) by L, we get

$$hL = Lx^2+v^2+dz^2, \qquad v = Ly+Mz. \tag{3.21}$$

By (3.121) and (3.131), $-d$ is a quadratic residue of L. Also, by assumption (3.13), $-L$ is a quadratic residue of d. Therefore† there exist integers λ, μ, η such that

$$L\lambda^2+\mu^2+d\eta^2 = 0, \qquad M\eta \equiv \mu \pmod{L} \tag{3.22}$$

with $L\lambda$, μ, $d\eta$ relatively prime in pairs. By (3.222) there exists an integer δ such that

$$\mu = L\delta+M\eta, \qquad (\delta,\eta) = 1. \tag{3.23}$$

Substitution of this value of μ into (3.221) yields

$$\phi(\delta,\eta) = -\lambda^2, \qquad (\delta,\eta) = 1. \tag{3.24}$$

* Oppenheim (**6**), 182.

† By a theorem due to Legendre. For exposition see Dickson (**3**), Ch. VIII (Statement on p. 130). See also Oppenheim (**6**), 179.

Hence there exists a transformation carrying ϕ into an equivalent form

$$\phi(y,z) = (-\lambda^2, M_1, N_1), \qquad M_1^2+N_1\lambda^2 = M^2-LN = -d. \quad (3.25)$$

Therefore f is equivalent to

$$x^2-\lambda^2y^2+2M_1yz+N_1z^2,$$

which by substitution of x for $x-\lambda y$ is equivalent to

$$x(x+2\lambda y)+2M_1yz+N_1z^2.$$

Since $(\lambda, d) = 1$, it follows from (3.252) that $(M_1, \lambda) = 1$, so that there exist integers k, l such that $\lambda l-M_1k = 1$. Hence the transformation

$$x_1 = \lambda x+M_1z, \qquad z_1 = kx+lz$$

is of determinant unity and yields the equivalent form

$$2x_1y+Ex_1^2+2Fx_1z_1+Gz_1^2.$$

Dropping subscripts and replacing y by $y+gx-Fz$, we obtain

$$h \sim 2xy+(E+2g)x^2-Hz^2,$$

where $H = d$ since the determinant is unchanged. Hence, if E is odd,

$$h \sim 2xy+x^2-dz^2 \quad (3.26)$$

by choice of g. If d is even, E is odd since h is properly primitive. Hence, to complete the proof of the lemma, it suffices to prove (3.26) for d odd and E even. In that case we choose $g = -\frac{1}{2}E$ and see that

$$h \sim 2xy-dz^2.$$

We replace x by $x+\frac{1}{2}(1-d)y$ and get $2xy+(1-d)y^2-dz^2$. Next we replace x by $x+dy+dz$ and z by $y+z$ and get (3.26).

4. Conditions (1.2) and (1.4).* If form (1.1) is primitive and non-classic, it represents all integers if and only if the improperly primitive classic form $f = 2\Psi$ represents all even integers. Therefore, henceforth we shall study a properly or improperly primitive classic form (2.2) and obtain necessary and sufficient conditions that it should represent all integers if properly primitive and all even integers if improperly primitive. Let Ω, Δ be the invariants of f, and let

$$F = Ax^2+By^2+Cz^2+2Ryz+2Sxz+2Txy \quad (4.11)$$

be its reciprocal. Multiplying both members of (2.2) by a and C in succession and completing squares at each step, we obtain first (2.4),

* See Oppenheim (6).

which becomes (2.5) in notations (2.3), and finally

$$aCf = CX^2+\Omega Y^2+\Omega\Delta az^2, \tag{4.12}$$

where $$X = ax+ty+sz, \qquad Y = Cy-Rz. \tag{4.13}$$

Next, in view of a well-known property of determinants,

$$Aa+Tt+Ss = \Omega\Delta, \quad Ta+Bt+Rs = 0, \quad Sa+Rt+Cs = 0. \tag{4.14}$$

Without loss of generality we may assume that either a or, in case f is improperly primitive, $\frac{1}{2}a$ is prime to $2\Omega\Delta$. Then the coefficients C, R, B of the binary form

$$(C, -R, B) = CY^2-2RYZ+BZ^2, \qquad BC-R^2 = \Delta a \tag{4.15}$$

have no odd prime factor in common. For if p divides all of C, R, B, it also divides both Δa, by (4.152), and the determinant $\Delta^2\Omega$ of F. Hence p is prime to a and by (4.14) it divides S, T, A, which is impossible since F is primitive. If a is odd, we see in the same way that 2 also does not divide all of C, R, B, and hence $(C, -R, B)$ is primitive.

Let m be represented by f. Then (2.5) holds with f replaced by m. Hence for every odd prime p dividing Ω,

$$am \equiv X^2 \pmod{p}.$$

If $(m, p) = 1$, then

$$(m|p) = (a|p), \tag{4.21}$$

and therefore f does not represent $\frac{1}{2}(p-1)$ classes modulo p of integers m which do not satisfy (4.21). Hence if an odd prime p divides Ω, neither f nor, in case f is improperly primitive, $\frac{1}{2}f$ is universal. If f is improperly primitive, Ω is odd and hence $\Omega = \pm 1$.

Next let f be properly primitive. Then a is odd and $(C, -R, B)$ is primitive. If Ω is divisible by 4, then, by (2.5), every odd integer m represented by f is congruent to a modulo 4, and hence f is not universal. Similarly is ruled out the case $\Omega = \pm 2$, $(C, -R, B)$ improperly primitive. Finally, consider a form f with $\Omega = \pm 2$, $(C, -R, B)$ properly primitive. Hence we may assume that C is odd. The equation (4.12) becomes

$$aCf = CX^2+2eY^2+2e\Delta az^2, \tag{4.22}$$

where $e = \Omega/|\Omega| = \pm 1$.

If f, and hence aCf, represents all odd residues modulo 8, then $Y^2+\Delta az^2 = \frac{1}{2}e(aCf-CX^2)$, with X odd, represents all residues modulo 4. This latter is impossible if Δ is odd or is a multiple of 4. It remains to consider $\Delta = 2\Delta_1$, Δ_1 odd.

If f is even, so is X. Write $X = 2X_1$. Then, by (4.22), if f represents all even residues modulo 16,

$$\tfrac{1}{2}aCf = 2CX_1^2 + eY^2 + e2\Delta_1 az^2$$

represents all and hence all odd residues modulo 8. If $\frac{1}{2}aCf$ is odd, so is Y, and $CX_1^2 + e\Delta_1 az^2$ represents all residues modulo 4, which is impossible since C and $e\Delta_1 a$ are both odd.

Next assume that f is properly primitive and that its determinant D is divisible by 4. By the above $\Omega = \pm 1$, $D = \Omega^2\Delta = \Delta$, and hence 4 divides Δ. Then since $(C, -R, B)$ is primitive, it is properly primitive by (4.152). Hence we may take C odd. By (4.12),

$$aCf \equiv CX^2 + \Omega Y^2 \pmod 4,$$

where $CX^2 + \Omega Y^2$ has but three incongruent residues modulo 4 since both C and Ω are odd. Therefore f is not universal. This proves that if f is universal, D is odd or double an odd integer.

Summing up: *If f is an improperly primitive form representing all even integers, its invariant $\Omega = \pm 1$. If f is a properly primitive form representing all integers, then its invariant $\Omega = \pm 1$ and its determinant $D = \Omega^2\Delta = \Delta$ is not divisible by 4. Since Ω is odd, the reciprocal F of f is properly primitive.*

5. Choice of a and C. Up to the present we have been avoiding the use of the following normalization.

LEMMA 3.* *If f is* (i) *a properly or* (ii) *an improperly primitive ternary quadratic form with a properly primitive reciprocal, then it is equivalent to a form f_1 such that* (i) *a_1 and C_1 are relatively prime and are prime to $2\Omega\Delta$, or* (ii) *$a_1 = 2\alpha$ and α and C_1 are relatively prime and are prime to $2\Omega\Delta$. Here a_1 is the leading coefficient of f_1, and C_1 is the third coefficient of the reciprocal F_1 of f_1.*

We shall outline a proof of it somewhat different from those referred to.

Since F is properly primitive it represents properly an integer prime to any assigned integer and hence to $2\Omega\Delta$. If $\Omega\Delta$ is odd, then F represents properly an integer $\equiv 1 \pmod 4$ and one $\equiv 3 \pmod 4$. Hence we may assume that C is prime to $2\Omega\Delta$ and, if $\Omega\Delta$ is odd,

$$C \equiv \Omega \pmod 4. \tag{5.11}$$

* Smith (8), §§ 5 and 9. Dickson (4), 15–17. Bachman (1), 64.

Then the binary form

$$\psi = ax^2+2txy+by^2 \qquad (ab-t^2 = \Omega C)$$

is primitive.

Similar to (4.14) we have

$$aS+tR+sC = 0, \quad tS+bR+rC = 0, \quad sS+rR+cC = \Omega\Delta. \quad (5.12)$$

The g.c.d. g of a, t, b divides $ab-t^2 = \Omega C$ and also the determinant $D = \Omega^2\Delta$ of f. Since C is prime to $2\Omega\Delta$, g divides Ω and is prime to C. Then by (5.121) and (5.122) g divides sC and rC, and hence s and r. Therefore, since g divides Ω, it divides cC and hence c, by (5.123). Thus g divides the g.c.d. of the coefficients $a,\ldots, t$ of f and, since f is primitive, $g = 1$. Hence ψ is primitive.

If f is properly primitive, so is ψ. The determinant of f is

$$abc+2rst-ar^2-bs^2-ct^2 = \Omega^2\Delta.$$

Let $\Omega^2\Delta$ be even. If ψ were improperly primitive, then c would be even by the above and f would be improperly primitive. If $\Omega\Delta$ is odd, $\Omega C \equiv 1 \pmod 4$ by (5.11), whereas if ψ were improperly primitive, we would have $\Omega C = ab-t^2 \equiv -1 \pmod 4$.

If f is improperly primitive, ψ is also improperly primitive since it is primitive.

If ψ is properly primitive it represents properly an integer a_1 prime to $2\Omega\Delta C$. If ψ is improperly primitive it represents properly an integer $a_1 = 2\alpha$ such that α is prime to $2\Omega\Delta C$. Hence ψ is equivalent to a form ψ_1 with a_1 as the leading coefficient. Transformation carrying ψ into ψ_1 carries f into $f_1 = a_1x^2+\ldots$. The contragredient (defined by the reciprocal matrix of its transpose) transformation, which carries F into the reciprocal F_1 of f_1, leaves C unaltered.* We take $C_1 = C$.

6. Characters of f. We may assume, by Lemma 3, that a and C or, in case f is improperly primitive, $\frac{1}{2}a$ and C are prime to $2\Omega\Delta$ and to one another. Multiplying both members of (4.11) by C and completing squares, we obtain

$$CF = (Cz+Ry+Sx)^2+(AC-S^2)x^2+2(CT-RS)xy+(BC-R^2)y^2. \quad (6.1)$$

In view of

$$AC-S^2 = \Delta b, \qquad RS-CT = t\Delta, \qquad BC-R^2 = \Delta a, \quad (6.2)$$

* See, for example, Dickson (**4**), 16.

equation (6.1) may be written as

$$CF = X^2+\Delta bx^2-2\Delta txy+\Delta ay^2, \qquad X = Cz+Ry+Sx. \quad (6.3)$$

Relations (2.5) and (6.3) may be used to enumerate all the generic characters of f. We shall here employ only the latter of the two to study characters of f relative to prime divisors of Δ, and, in case f is improperly primitive, also characters with respect to 4 and 8.

Let M be represented by F. Then (6.3) holds with F replaced by M. If p is an odd prime dividing Δ, then for every M prime to p, $MC \equiv X^2 \pmod{p}$. Hence

$$(M|p) = (C|p),$$

and f possesses character $(F|p)$. If f is improperly primitive, Δ is even and so are a and b. Let M be odd. Then, by (6.3),

$$CM \equiv X^2 \equiv 1 \pmod{4}, \quad (6.4)$$

and therefore f has the character $(-1)^{\frac{1}{4}(F-1)}$. If Δ is divisible by 4, the congruence (6.4) holds modulo 8 and hence f has also the character $(-1)^{\frac{1}{4}(F^2-1)}$.

7. Conditions (1.3), (1.5), and (1.6). Let p be an odd prime dividing Δ. Let $f = pm$. Then

$$CX^2+\Omega Y^2 \equiv 0 \pmod{p}$$

in (4.12). If $(C|p) = -(-\Omega|p)$, $X \equiv Y \equiv 0 \pmod{p}$, and upon division by p (4.12) yields

$$Cm \equiv \Omega\Delta z^2 \pmod{p},$$

whereas f represents all multiples of p. Therefore $(C|p) = (-\Omega|p)$. But $(F|p) = (C|p)$. Hence (1.3) and (1.52) hold.

If f is improperly primitive,

$$\Omega C = ab-t^2 \equiv -1 \pmod{4}, \quad (7.11)$$

since Ω and C are odd. But $(-1)^{\frac{1}{4}(F-1)} = (-1)^{\frac{1}{4}(C-1)}$, and hence (1.51) holds. Next let Δ be divisible by 4. In order that f should represent all even integers, aCf must represent all multiples of 4 and hence all multiples of 8 modulo any power of 2.

If Δ is divisible by 8, the only residues of aCf modulo 16 are those of $CX^2+\Omega Y^2$ by (4.12). If

$$C \equiv -\Omega+4 \pmod{8}, \quad (7.21)$$

$aCf \not\equiv 8 \pmod{16}$. For if $CX^2+\Omega Y^2 \equiv 8 \pmod{16}$ and X and Y are odd, then $0 \equiv CX^2+\Omega Y^2 \equiv C+\Omega \pmod{8}$, which contradicts (7.21). Hence $X \equiv Y \equiv 0 \pmod{2}$. Write $X = 2X_1$, $Y = 2Y_1$.

Then $CX_1^2+\Omega Y_1^2 \equiv 2 \pmod 4$, which is impossible since, by (7.11), $CX_1^2+\Omega Y_1^2 \equiv C(X_1^2-Y_1^2) \pmod 4$. Hence, in view of (7.11), $C \equiv -\Omega \pmod 8$.

If $\Delta \equiv 4 \pmod 8$, the only residues of $\Omega\Delta az^2$ modulo 64 are 0, $\Omega\Delta a$, 32, where $\Omega\Delta a \equiv 8 \pmod{16}$. If (7.21) holds, we show as above that $CX^2+\Omega Y^2 \not\equiv 8, 24, 32, 40, 56 \pmod{64}$. Then the only possible residues modulo 64 which are $\equiv 0 \pmod 8$ are 0, 16, 48, and those obtained by adding to them 32 and $\Omega\Delta a$. Hence the residue $32+\Omega\Delta a$ is missing. Therefore, as before, $C \equiv -\Omega \pmod 8$ and (1.6) holds.

8. Sufficiency of the conditions (1.2), (1.3), and (1.4)–(1.6). Since a can be taken as any integer represented by the properly or improperly primitive binary form $\psi = ax^2+2txy+by^2$, we may assume that a is (i) an odd prime or (ii) the double of an odd prime not dividing $\Omega\Delta a$ according as f is (i) properly or (ii) improperly primitive.* Here, as before, C is prime to $2\Omega\Delta$. Therefore, by Lemma 1, f represents (i) all integers, or (ii) all even integers, if and only if

$$\Phi(X,y,z) = X^2+\Omega Cy^2-2\Omega Ryz+\Omega Bz^2 \tag{8.11}$$

represents (i) all multiples of a, or (ii) all even multiples of a. Here, as before, since C is prime to $2\Omega\Delta$,

$$(F|p) = (C|p) \tag{8.121}$$

for every odd prime p dividing Δ and, if f is improperly primitive, also

$$(-1)^{\frac{1}{2}(F-1)} = (-1)^{\frac{1}{2}(C-1)}, \qquad (-1)^{\frac{1}{2}(F^2-1)} = (-1)^{\frac{1}{2}(C^2-1)}. \tag{8.122}$$

By (8.121) and (1.3) in the case (i), and (1.52) in the case (ii), $-\Omega C$ is a quadratic residue of every odd prime divisor p of $\Delta = \Omega^2\Delta = D$.

Case (i). Here f is properly primitive, whence D is not divisible by 4, by (1.21), and the above implies that $-\Omega C$ is a quadratic residue of D. By (2.3), $-\Omega C$ is also a quadratic residue of a. Hence, since a is odd, $-\Omega C$ is a quadratic residue of $d = Da$ and we can apply Lemma 2 with $L = \Omega C$, $M = -\Omega R$, $N = \Omega B$. Therefore form (8.11) is equivalent to

$$\Phi_1 = 2xy+x^2-dz^2. \tag{8.21}$$

For $z = 0$, $x = 1$ or 2, $\Phi_1 = 2y+1$ or $4(y+1)$. Next, if d is even, whence $d \equiv 2 \pmod 4$, take $z = 1$, $x = 2$. Then $\Phi_1 = 4(y+1)+d$. If d is odd, take $z = 1$, $x = 1$, and $y = 2u$ or $2u+1$ according as

* Weber (**9**).

$d = 4h-1$ or $4h+1$. Then $\Phi_1 = 4(u-h)+2$. In both cases Φ_1 is equal to any $4l+2$ by choice of y and u respectively. Thus Φ_1 and hence form (8.11) represent all integers. Therefore (8.11) represents all multiples of a, and hence f is universal. This completes the proof of Theorem 1.

Case (ii). In this case $a = 2q$, where q is an odd prime. Write $D = \Delta = 2^k\Delta_1$, where Δ_1 is odd. Since D is even, $k \geqslant 1$. The determinant d of the auxiliary form (8.11) is $Da = 2^{k+1}\Delta_1 q$. As before, by (2.3), $-\Omega C$ is a quadratic residue of a and hence of q. By (1.52), $-\Omega C$ is a residue of every odd prime factor p of Δ_1. Hence $-\Omega C$ is a quadratic residue of $\Delta_1 q$. It remains to show that $-\Omega C$ is a quadratic residue of 2^{k+1}.

If $k = 1$, (1.51) and (8.1221) imply that $C \equiv -\Omega \pmod 4$, whence $-\Omega C \equiv 1 \pmod 4$, and therefore $-\Omega C$ is a quadratic residue of $4 = 2^{k+1}$.

If $k \geqslant 2$, it suffices to prove that $-\Omega C$ is a quadratic residue of 8. As in the preceding case,

$$-\Omega C \equiv 1 \pmod 4. \tag{8.31}$$

Since $\Omega = \pm 1$, (1.6) and (8.1222) imply that $C \equiv \pm\Omega$ and hence $-\Omega C \equiv \mp 1 \pmod 8$. Therefore $-\Omega C \equiv 1 \pmod 8$ by (8.31) and $-\Omega C$ is a quadratic residue of 8.

Thus $-\Omega C$ is a quadratic residue of $d = 2^{k+1}\Delta_1 q$, and we can apply Lemma 2 with $L = \Omega C$, $M = -\Omega R$, $N = \Omega B$. Therefore form (8.11) is equivalent to

$$2yz+z^2-dx^2. \tag{8.32}$$

Take $x = 0$ and $z = 2$. Then (8.32) becomes $4(y+1)$ which can be identified with any multiple $4l$ of 4 by choice of y. Therefore (8.32) represents all multiples of 4 and hence all even multiples of a. Then by Lemma 1, $f = 2f_1$ represents all even integers. This proves Theorem 2.

9. Universal ternary forms are zero forms. A quadratic form is said to be a zero form if and only if it represents zero for the values of the variables not all zero. When one compares Theorems 1 and 2 with the general criterion of Smith* for the existence of non-zero solutions of $f = 0$, one arrives at the following:

* Smith (8), 410.

THEOREM 3. *Every universal, classic or non-classic, ternary quadratic form is a zero form.*

This theorem was first conjectured and proved by Dickson.* We shall give yet another, an elementary, proof of this theorem. As before, in place of a universal non-classic form f_1, we shall consider the improperly primitive classic form $f = 2f_1$ representing all even integers.

Write $\Delta = \Delta_1 \Delta_2^2$, where Δ_2^2 is the largest square dividing Δ. If f is universal, it represents $-\Delta_1$. Since $-\Delta_1$ has no square factors, f represents it properly. Hence f is equivalent to a form with $a = -\Delta_1$. If f is improperly primitive and represents all even integers, it represents properly $-\Delta_1$ if this latter is even, and $-4\Delta_1$ if $-\Delta_1$ is odd. For, if $-\Delta_1$ is odd, then $-2\Delta_1$ is free from square factors and is represented, and hence properly represented, by the universal non-classic form $f_1 = \frac{1}{2}f$. Therefore, if f is improperly primitive, we may take $a = -\Delta_1$ or $-4\Delta_1$. Thus in both cases (4.12) becomes

$$aCf = CX^2 + \Omega(Y^2 - \alpha^2 z^2) = G(X, Y, z), \tag{9.1}$$

where $\alpha = \Delta_1 \Delta_2$ or $2\Delta_1 \Delta_2$. We see at once that $X = 0, y = \alpha, z = 1$ satisfy the equation $G = 0$. Therefore by (4.13) there exist rational and therefore integral solutions not all zero of the equation $f = 0$. To complete the proof it suffices to note that f_1 is a zero form if and only if $f = 2f_1$ is such.

Although the comparison of the Theorems 1 and 2 and the above-mentioned criterion of Smith shows that there are zero forms which are not universal, it permits us to state the following theorem for the case of determinants free from square factors.

THEOREM 4. *Every zero, ternary, quadratic form f of quadratfrei determinant is universal if it is properly primitive and represents all even integers if improperly primitive.*

We can also prove it simply as follows. Since f is a zero form it is equivalent to a form f_1 with the leading coefficient zero. Write

$$f_1 = by^2 + cz^2 + 2ryz + 2sxz + 2txy,$$

$s = gs_1$, $t = gt_1$, $g = (s, t)$. There exist integers σ and τ such that $t_1\tau - s_1\sigma = 1$. Transformation $y_1 = t_1 y + s_1 z$, $z_1 = \sigma y + \tau z$ carries f_1 into a form f' with $a' = s' = 0$. Hence, omitting the accents,

$$f \sim by^2 + cz^2 + 2ryz + 2txy.$$

* Dickson (**4**), 17.

The determinant of the latter form is $D = -t^2c$. Since D is free from square factors, we may take $t = 1$.* Thus

$$f \sim by^2 - Dz^2 + 2ryz + 2xy.$$

Replace x by $x+ky-rz$. Then, by choice of k, f is equivalent to either

$$2xy+y^2-Dz^2 \quad \text{or} \quad 2xy-Dz^2 \tag{9.2}$$

according as b is odd or even, and hence is universal if properly primitive. If f is improperly primitive, $b \equiv D \equiv 0 \pmod 2$ and f is equivalent to (9.22), and hence represents all even integers.

* If $t = -1$, we change the signs of x and z.

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LAPLACE'S OPERATOR IN CURVILINEAR COORDINATES

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In this note I give, by methods of direct differentiation, a simple transformation of Laplace's operator ∇^2 to general curvilinear co-ordinates u, v, w. Let us denote partial derivatives and partial differential operators by suffixes, thus u_x, ∂_u, and let $\sum$ always denote summation over x, y, z. Then we may start with the transformation partially effected in the form*

$$\nabla^2 = (A, B, C, F, G, H)(\partial_u, \partial_v, \partial_w)^2 + (\nabla^2 u)\partial_u + (\nabla^2 v)\partial_v + (\nabla^2 w)\partial_w, \quad (1)$$

where $$A \equiv \sum u_x^2, \qquad F \equiv \sum v_x w_x, \text{ etc.}$$

To evaluate $\nabla^2 u$ we regard the three equations

$$\begin{aligned} u_x^2 + u_y^2 + u_z^2 &= A, \\ u_x v_x + u_y v_y + u_z v_z &= H, \\ u_x w_x + u_y w_y + u_z w_z &= G, \end{aligned}$$

as linear equations in u_x, u_y, u_z. So solving them we get

$$u_x = \frac{A}{J}\frac{\partial(v,w)}{\partial(y,z)} + \frac{H}{J}\frac{\partial(w,u)}{\partial(y,z)} + \frac{G}{J}\frac{\partial(u,v)}{\partial(y,z)}, \text{ etc.}, \quad (2)$$

where $$J = \frac{\partial(u,v,w)}{\partial(x,y,z)}.$$

Differentiate these equations (2) in x, y, z respectively and add to give $\nabla^2 u$. In this sum the coefficient of A/J will be

$$\sum \begin{vmatrix} v_{xy} & w_{xy} \\ v_z & w_z \end{vmatrix} + \sum \begin{vmatrix} v_y & w_z \\ v_{zx} & w_{zx} \end{vmatrix},$$

which vanishes, since the determinants cancel in pairs. Thus A/J, H/J, G/J do not appear in undifferentiated form in the expression for $\nabla^2 u$, and we are left with

$$\nabla^2 u = \sum \frac{\partial(v,w)}{\partial(y,z)}\frac{\partial}{\partial x}\left(\frac{A}{J}\right) + \sum \frac{\partial(w,u)}{\partial(y,z)}\frac{\partial}{\partial x}\left(\frac{H}{J}\right) + \sum \frac{\partial(u,v)}{\partial(y,z)}\frac{\partial}{\partial x}\left(\frac{G}{J}\right).$$

But $$\frac{\partial(v,w)}{\partial(y,z)} = J\frac{\partial x}{\partial u},$$

* Compare, for instance, Goursat, *Cours d'analyse* (1910), i. 156 (69), which is the corresponding equation for orthogonal coordinates.

and so $$\sum \frac{\partial(v,w)}{\partial(y,z)}\frac{\partial}{\partial x} = J\sum \frac{\partial x}{\partial u}\frac{\partial}{\partial x} = J\frac{\partial}{\partial u}.$$

Thus $$\nabla^2 u = J\left\{\frac{\partial}{\partial u}\left(\frac{A}{J}\right)+\frac{\partial}{\partial v}\left(\frac{H}{J}\right)+\frac{\partial}{\partial w}\left(\frac{G}{J}\right)\right\}.$$

Substitution in (1) and reduction gives

$$\frac{1}{J}\nabla^2 = \partial_u\left(\frac{A}{J}\partial_u+\frac{H}{J}\partial_v+\frac{G}{J}\partial_w\right)+\partial_v\left(\frac{H}{J}\partial_u+\frac{B}{J}\partial_v+\frac{F}{J}\partial_w\right)+$$
$$+\partial_w\left(\frac{G}{J}\partial_u+\frac{F}{J}\partial_v+\frac{C}{J}\partial_w\right). \quad (3)$$

This formula (3) still holds, if we write

$$\nabla^2 \equiv (a,b,c,f,g,h)(\partial_x,\partial_y,\partial_z)^2,$$
$$A \equiv (a,b,c,f,g,h)(u_x,u_y,u_z)^2, \text{ etc.},$$
$$2H \equiv v_x\frac{\partial A}{\partial u_x}+v_y\frac{\partial A}{\partial u_y}+v_z\frac{\partial A}{\partial u_z}, \text{ etc.},$$

provided that the coefficients a, b, c, f, g, h are constants. The generalization to many dimensions is obvious.